

On Geometry Of Some Curve Pairs According To Type Of Bishop Frame In Lorentz 3-Space

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Abstract

The aim of this study is to investigate the associated curves of the normal indicatrix of a non-null curve in Lorentz 3-space. To achieve this goal, firstly, the vectors used in the Frenet elements of the normal indicatrix is utilized to create version of the Bishop frame by taking advantage of the rotations around them. The relationships between the elements of the obtained frames and the associated curves of the normal indicatrix is examined. Additionally, the similarities of these associated curves to the evolute, Mannheim, and Bertrand curve pairs, which are commonly encountered in differential geometry, are studied.

Keywords: Type of Bishop frame, evolute curve, Bertrand curve, Mannheim curve, Lorentz space.

I. INTRODUCTION

In theoretical physics, the Lorentz space has different properties compared to Euclidean space. The most important of these differences is that while there are only spacelike dimensions in Euclidean space, the Lorentz space has a timelike dimension. Considering these differences, the differential geometry of curves in the Lorentz space contains more general and striking results compared to Euclidean space. One of the most natural methods when comparing two curves is to use Frenet elements at the corresponding points. Curves with the same properties at corresponding points are called associated curves or curve pairs. Examining the geometric relationship between two curves is extremely important. In this way, it is possible to learn about one curve by starting from the properties of the other. The most associated curve pairs are involute-evolute curves, Mannheim curves, and Bertrand curves. These curve pairs have a wide range of applications from physics to kinematics.

Many studies have been done on curves and their characterizations. The relationships between the curves and associated curves have been examined using Frenet frame fields. Later, these relationships were studied in fields such as Lorentz space, Galilean space and dual space, which have different properties than Euclidean space. The results obtained also receive interest in surface and manifold theories. The curve pairs that have been most studied on geometry are involute-evolute curves, Bertrand curves and Mannheim curves. Involute-evolute curve pairs discovered by C. Huggens are defined as curves that have normal vectors perpendicular to the tangent vectors at every point (Hacısalıhoğlu, 1983). Frenet frame fields and some important relations have been obtained in Euclidean 3-space and higher dimensional spaces for this curve pair. For example, Turgut and Erdogan (1992) studied the curvature theory of involute-evolute curve pairs in n-dimensional Euclidean space and gave important results related to horizontal lifts. Studies on involute-evolute curves are not limited to Euclidean space. There are studies in which the counterpart of the results obtained in Euclidean space is investigated in Lorentz space (Bilici and Çalışkan, 2009).

Bertrand curves were first defined by Bertrand Russell in 1850 as curves with parallel principal normals. These curves, which have many application areas, have been studied using different structures to determine their geometric properties and many characterizations have been obtained (Izumiya and Takeuchi, 2002; Matsuda and Yorozu, 2003; Cheng 2009; Babaarslan and Yaylı, 2013). If the principal normal vector field of one curve is equal to the binormal vector field of the other curve at corresponding points, these curves are called Mannheim curve pairs. Many research articles have been written on these curves (Liu, 2008, Ozkaldı et al., 2009, Lone et al., 2019). On the other hand, in 1975 Bishop introduced the Bishop frame or parallel frame field as an alternative to the

Frenet frame field (Bishop, 1975). This frame field is used as an alternative frame in cases where the Frenet frame is not defined, that is, when the second derivative of the curve is zero. Important expansions of the curvature theory obtained with the Frenet frame were re-examined and geometric interpretations were developed with the help of the Bishop frame. In addition, structures called Bishop-type frames were created by using the method of obtaining this frame field. These structures were later carried to many different geometric structures, especially Lorentz space (Bükcü and Karacan, 2008; Grbovic and Nesovic, 2018; Turhan, 2020; Yılmaz, 2021).

II. MATERIALS AND METHOD

This section recalls some fundamental formulas and notions for later use. Lorentz 3-space is an important area of study for those involved in theoretical physics and mathematics. Elements in Lorentz space are grouped into three categories: spacelike, timelike, and lightlike. Lorentz 3-space is a 3-dimensional space with a specific geometric structure defined by the Lorentz metric, which is different from the Euclidean metric. We denote by E_1^3 Lorentz 3-space with the scalar product

$$\langle u, v \rangle = -u_1v_1 + u_2v_2 + u_3v_3$$

and the vector product

$$u \times v = \left(-\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \begin{vmatrix} u_3 & u_1 \\ v_3 & v_1 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right),$$

where $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ are two vectors of E_1^3 .

The vector $v \in E_1^3$ is said to be spacelike, lightlike or timelike whenever $\langle v, v \rangle > 0$ or $v = 0$, $\langle v, v \rangle = 0$ and $v \neq 0$ and $\langle v, v \rangle < 0$, respectively. Similarly, an arbitrary curve $\gamma = \gamma(s)$ can be spacelike, lightlike or timelike, if all of its velocity vectors $\gamma'(s)$ are, respectively, spacelike, lightlike or timelike, [16, 20].

Let $\gamma(s)$ be an arbitrary spacelike curve and $\{t, n, b\}$ is the Frenet frame along the $\gamma(s)$ in E_1^3 . The Frenet frame for this curve is given as

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \epsilon\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix},$$

where $\epsilon = \pm 1$ and the functions κ, τ are curvature and torsion of the $\gamma(s)$, respectively, [12]. If $\epsilon = 1$, then $\gamma(s)$ is a spacelike curve with timelike principal normal and spacelike binormal. Also, if $\epsilon = -1$, then $\gamma(s)$ is a spacelike curve with spacelike principal normal and timelike binormal.

Definition 2.1. Let α and β be two regular curves in Lorentz 3-space, with the Frenet frame $\{t_\alpha, n_\alpha, b_\alpha\}$ and $\{t_\beta, n_\beta, b_\beta\}$, respectively.

- i. β is known the evolute curve of α if and only if $n_\beta = t_\alpha$.
- ii. β is known the Bertrand mate curve of α if and only if $n_\beta = n_\alpha$.
- iii. β is known the Mannheim curve of α if and only if $n_\beta = b_\alpha$, [5,15].

Definition 2.2. Let γ be a spacelike curve in the Lorentz 3-space. If we translate of the principal normal vector field to the center of unit pseudo-sphere S_1^2 , we obtain a pseudo-spherical curve $\gamma_n(s_n) = n(s)$. This curve is called normal indicatrix of the curve $\gamma = \gamma(s)$.

III. DIRECTION CURVES OF NORMAL INDICATRIX OF SPACELIKE CURVES WITH TIMELIKE PRINCIPAL NORMAL

Let $\gamma = \gamma(s)$ be a spacelike curve with timelike principal normal in Lorentz 3-space with Frenet frame $\{t, n, b\}$ and $\gamma_n = \gamma_n(s_n)$ be its normal indicatrix with $\{t_n, n_n, b_n\}$. For the normal indicatrix γ_n , consider a vector field

$$X(s_n) = f(s_n)t_n(s_n) + g(s_n)n_n(s_n) + h(s_n)b_n(s_n) \quad (1)$$

where s_n is arc-length parameter of γ_n and f, g, h are real functions. We take into consideration only a unit spacelike vector field X for our calculations, so we obtain:

$$f^2(s_n) - g^2(s_n) + h^2(s_n) = 1. \quad (2)$$

If we get differentiation of (2), we have

$$f(s_n)f'(s_n) - g(s_n)g'(s_n) + h(s_n)h'(s_n) = 0.$$

So, the definition of X -direction curve of the spacelike normal indicatrix γ_n of a curve γ is given as follows:

Definition 3.1. Let γ be a spacelike curve with timelike principal normal in the Lorentzian 3-space, γ_n be the normal indicatrix of a curve γ and X be the unit spacelike vector field satisfies equations (1) and (2). The integral curve of X is called X -direction curve of γ_n , [15, 19].

Let $\beta = \beta(s)$ be a X -direction curve of γ_n with Frenet apparatus $\{t_\beta, n_\beta, b_\beta, \kappa_\beta, \tau_\beta\}$. Then, from above definition, we get

$$\frac{d\beta}{ds_n} = X(s_n)$$

or

$$\frac{d\beta}{ds_\beta} \frac{ds_\beta}{ds_n} = X(s_n),$$

where s_β is the arc-length parameter of β .

Now, we assume that $s_\beta = s_n$. Then, by differentiating (1) and use definition of X -direction curve, we obtain the following equation

$$\kappa_\gamma n_\gamma = (f' + y\kappa_n)t_n + (g' + f\kappa_n + h\tau_n)n_n + (h' + g\tau_n)b_n. \quad (3)$$

By using (3), we obtain and present new direction curves such as evolute-direction curves, Bertrand-direction curves and Mannheim-direction curves of the normal indicatrix γ_n . Also, we get some properties related with these curves.

Definition 3.2. Let γ be a spacelike curve with timelike principal normal in the Lorentzian 3-space, γ_n be the normal indicatrix of a curve γ with respect to Frenet apparatus $\{t_n, n_n, b_n, \kappa_n, \tau_n\}$. If we rotate the Frenet frame around the $t_n(s_n)$ -axis up to $\phi(s_n)$, we obtain the frame $\{t_n, (m_1)_n, (m_2)_n\}$ as follows;

$$\begin{bmatrix} t_n \\ n_n \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\phi(s_n) & -\sinh\phi(s_n) \\ 0 & -\sinh\phi(s_n) & \cosh\phi(s_n) \end{bmatrix} \begin{bmatrix} t_n \\ (m_1)_n \\ (m_2)_n \end{bmatrix}. \quad (4)$$

This frame is known as the type-1 of Bishop frame of normal indicatrix γ_n , [19].

In the following theorem, we give characterization of the X -direction curves of the normal indicatrix γ_n of γ which are evolute of γ_n , [19].

Theorem 3.1. Let γ be a spacelike curve with timelike principal normal, γ_n be normal indicatrix of γ with respect to $\{t_n, n_n, b_n, \kappa_n, \tau_n\}$ and β be X -direction curve of γ_n . If we rotate the Frenet frame in E_1^3 around the t_n -axis up to $\phi(s_n) = -\int_{\gamma_n} \tau_n(s_n) ds_n$, element of version of the type-1 Bishop frame $(m_2)_n$ is the evolute curves of γ_n .

Definition 3.3. Let γ be a spacelike curve with timelike principal normal in the Lorentzian 3-space, γ_n be the normal indicatrix of a curve γ with respect to Frenet apparatus $\{t_n(s_n), n_n(s_n), b_n(s_n)\}$. If we rotate the Frenet frame around the $n_n(s_n)$ -axis up to $\phi(s_n)$, we obtain the frame $\{(l_1)_n(s_n), n_n(s_n), (l_2)_n(s_n)\}$ as follows;

$$\begin{bmatrix} t_n(s_n) \\ n_n(s_n) \\ b_n(s_n) \end{bmatrix} = \begin{bmatrix} \cos\phi(s_n) & 0 & \sin\phi(s_n) \\ 0 & 1 & 0 \\ -\sin\phi(s_n) & 0 & \cos\phi(s_n) \end{bmatrix} \begin{bmatrix} (l_1)_n(s_n) \\ n_n(s_n) \\ (l_2)_n(s_n) \end{bmatrix} \quad (5)$$

This frame is known as the type-2 of Bishop frame of normal indicatrix γ_n .

Now, in the following theorem, we present a characterization of the X-direction curves of γ_n which are Bertrand of γ_n . Then, we give some relations between these curves.

Theorem 3.2. Let γ be a spacelike curve with timelike principal normal, γ_n be normal indicatrix of γ with respect to $\{t_n, n_n, b_n, \kappa_n, \tau_n\}$ and β be X-direction curve of γ_n . If we rotate the Frenet frame around the n_n -axis up to $\Phi(s_n) = -\phi$ constant angle, elements of the type-2 of Bishop frame $(l_1)_n$ and $(l_2)_n$ are the Bertrand curves of γ_n .

Proof. Let β be X-direction curve of γ_n and β be its Bertrand mate. By using (3), we get the following differential equations system:

$$\begin{aligned} f' + g\kappa_n &= 0, \\ g' + f\kappa_n + h\tau_n &= \kappa_\beta, \\ h' + g\tau_n &= 0. \end{aligned} \quad (6)$$

If we multiply the first, the second and the third equation in (6) with f , g and h respectively, and make necessary algebraic operations, we get $g = 0$. If we use the value of g in (6), we have $f = c_1$ and $h = c_2$, where c_1, c_2 are real constants. Because of $f^2 + h^2 = 1$, we obtain

$$f = \cos\phi, \quad h = \sin\phi \quad \text{veya} \quad f = -\sin\phi, \quad h = \cos\phi,$$

where ϕ is a constant angle between the spacelike tangent vector of the curve γ_n and the spacelike tangent vector of the curve β . Hence, integral curves are called Bertrand curves of γ_n as follows

$$\begin{cases} X_1(s_n) = \cos\phi t_n + \sin\phi b_n, \\ X_2(s_n) = -\sin\phi t_n + \cos\phi b_n. \end{cases} \quad (7)$$

On the other hand, from (5), we have

$$(l_1)_n(s_n) = \cos\phi(s_n)t_n - \sin\phi(s_n)b_n,$$

$$(l_2)_n(s_n) = \sin\phi(s_n)t_n + \cos\phi(s_n)b_n.$$

For $\phi(s_n) = -\phi$ constant angle,

$$\begin{cases} (l_1)_n(s_n) = \cos\phi t_n + \sin\phi b_n, \\ (l_2)_n(s_n) = -\sin\phi t_n + \cos\phi b_n. \end{cases} \quad (8)$$

So, from (7) and (8), we can easily say that $(l_1)_n$ and $(l_2)_n$ curves are the Bertrand curves of γ_n .

Definition 3.4. Let γ be a spacelike curve with timelike principal normal in E_1^3 , γ_n be the normal indicatrix of a curve γ with respect to Frenet frame $\{t_n(s_n), n_n(s_n), b_n(s_n)\}$. If we rotate the Frenet frame around the $b_n(s_n)$ -axis up to $\phi(s_n)$, we have the frame $\{(k_1)_n(s_n), (k_2)_n(s_n), b_n(s_n)\}$ as follows;

$$\begin{bmatrix} t_n(s_n) \\ n_n(s_n) \\ b_n(s_n) \end{bmatrix} = \begin{bmatrix} \sinh\phi(s_n) & \cosh\phi(s_n) & 0 \\ -\cosh\phi(s_n) & -\sinh\phi(s_n) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (k_1)_n(s_n) \\ (k_2)_n(s_n) \\ b_n(s_n) \end{bmatrix}. \quad (9)$$

This frame is known as the type-3 of Bishop frame of normal indicatrix γ_n .

Now, in the following theorem, we obtain a characterization of the X-direction curves of γ_n which are Mannheim of γ_n . Then, we give some relations between these curves.

Theorem 3.3. Let γ be a spacelike curve with timelike principal normal, γ_n be normal indicatrix of γ with respect to $\{t_n, n_n, b_n, \kappa_n, \tau_n\}$ and β be X-direction curve of γ_n . If we rotate the Frenet frame around the b_n -axis up to $\Phi(s_n) = \int_{\gamma_n} \kappa_n(s_n) ds_n$, elements of the type-3 of Bishop frame $(k_2)_n$ is the Mannheim curve of γ_n .

Proof. Let β be X-direction curve of γ_n and β be its Mannheim mate. By using (3), we get the following differential equations system:

$$\begin{aligned} f' + g\kappa_n &= 0, \\ g' + f\kappa_n + h\tau_n &= 0, \\ h' + g\tau_n &= \kappa_\beta. \end{aligned} \quad (10)$$

If we multiply the first, the second and the third equation in (10) with f , g and h respectively, and make necessary calculations, we get $h = 0$. If we use the value of h in (10), we obtain,

$$f = \cosh\phi(s_n) \text{ and } g = \sinh\phi(s_n). \quad (11)$$

If we use $h = 0$ and (11) in (10), we obtain

$$\phi(s_n) = \int_{\gamma_n} \kappa_n(s_n) ds_n.$$

Then, the vector coordinates of $X(s_n)$ are written by

$$h(s_n) = 0, f(s_n) = \cosh(\int \kappa_n(s_n) ds_n) \text{ and } g(s_n) = \sinh(\int \kappa_n(s_n) ds_n).$$

Hence, integral curves are called Mannheim curves of γ_n as follows

$$X(s_n) = \cosh(\int \kappa_n(s_n) ds_n) t_n(s_n) + \sinh(\int \kappa_n(s_n) ds_n) n_n(s_n). \quad (12)$$

Also, from (9), we have

$$\begin{aligned} (k_1)_n(s_n) &= -\sinh\phi(s_n)t_n - \cosh\phi(s_n)n_n, \\ (k_2)_n(s_n) &= \cosh\phi(s_n)t_n + \sinh\phi(s_n)n_n. \end{aligned}$$

For $\phi(s_n) = \int \kappa_n(s_n) ds_n$, we obtain,

$$(k_2)_n(s_n) = \cosh(\int \kappa_n(s_n) ds_n) t_n(s_n) + \sinh(\int \kappa_n(s_n) ds_n) n_n(s_n). \quad (13)$$

So, from (12) and (13), we can say that $(k_2)_n(s_n)$ is the Mannheim curve of γ_n .

Example 3.1. Consider a spacelike curve γ

$$\gamma(s) = \left(\frac{s}{2}, \frac{\sqrt{3}}{2} \sinh s, \frac{\sqrt{3}}{2} \cosh s \right).$$

Let $n(s)$ be the timelike principal normal vector field of γ as

$$n(s) = (0, \sinh s, \cosh s).$$

Using $s = s_n$, the normal indicatrix of γ is given as

$$\gamma_n(s_n) = (0, \sinh(s_n), \cosh(s_n))$$

If we make necessary algebraic operations, the Frenet apparatus of $\gamma_n(s_n)$ is given as follows;

$$t_n(s_n) = (0, \cosh(s_n), \sinh(s_n)),$$

$$n_n(s_n) = (0, \sinh(s_n), \cosh(s_n)),$$

$$b_n(s_n) = (1, 0, 0),$$

$$\kappa_n(s_n) = 1,$$

$$\tau_n(s_n) = 0.$$

In Fig.1 and Fig.2, the graphs of the $\gamma(s)$ and $\gamma_n(s_n)$ are given, respectively.

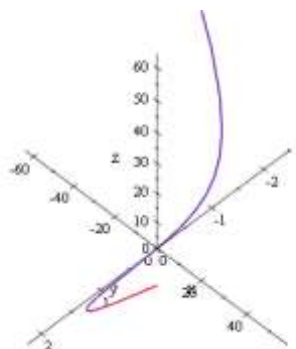


Figure 1. The curve $\gamma(s)$

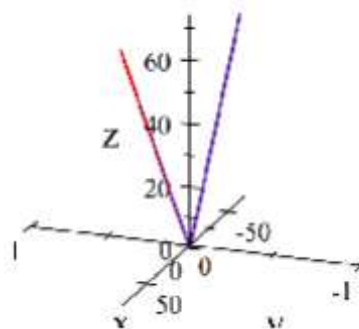


Figure 2. The normal indicatrix $\gamma_n(s_n)$

Let $(m_2)_n$, $(l_1)_n$ and $(k_2)_n$ be evolute curve, Bertrand curve and Mannheim curve of $\gamma_n(s_n)$ of $\gamma(s)$, respectively. From (4), (5) and (9), we have

$$(m_2)_n = (\cosh \phi, \sinh \phi \sinh s_n, \sinh \phi \cosh s_n),$$

$$(l_1)_n = (\sin \phi, \cos \phi \cosh s_n, \cos \phi \sinh s_n),$$

$$(k_2)_n = (0, \cosh \phi \cosh s_n + \sinh \phi \sinh s_n, \cosh \phi \sinh s_n + \sinh \phi \cosh s_n).$$

On the other hand, the integral curve of the vector field $(m_2)_n$ is known evolute direction curve, [19]. Similarly, the integral curves of $(l_1)_n$ and $(k_2)_n$ are called Bertrand direction curve and Mannheim direction curve, respectively. These direction curves given as,

$$\beta(m_2)_n = (\cosh \phi s_n + c_1, \sinh \phi \cosh s_n + c_2, \sinh \phi \sinh s_n + c_3),$$

$$\beta(l_1)_n = (\sin \phi s_n + c_4, \cos \phi \sinh s_n + c_5, \cos \phi \cosh s_n + c_6),$$

$$\beta(k_2)_n = (c_7, \cosh \phi \sinh s_n + \sinh \phi \cosh s_n + c_8, \cosh \phi \cosh s_n + \sinh \phi \sinh s_n + c_9).$$

If we get $\phi = \frac{\pi}{3}$ and $c_i = 1$, ($i = 1, \dots, 9$) in equations of direction curves, we have the $\beta(m_2)_n$, $\beta(l_1)_n$ and $\beta(k_2)_n$ direction curves as illustrated in Fig. 3 and Fig. 4.

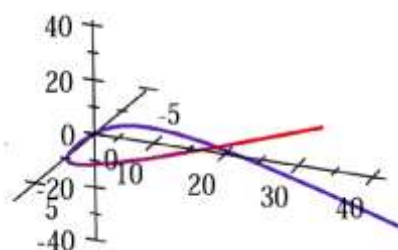


Figure 3. Evolute/Bertrand direction curve of $\gamma_n(s_n)$

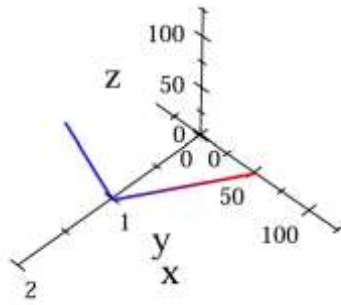


Figure 4. Mannheim direction curve of $\gamma_n(s_n)$

VI. REFERENCES

- [1] İlarıslan, K., & Aslan, N.K. (2017). On Spacelike Bertrand Curves in Minkowski 3-Space. *Konuralp Journal of Mathematics*, 5(1), 214-222.
- [2] Bukcu, B., & Karacan, M. K. (2007). On the involute and evolute curves of the spacelike curve with a spacelike binormal in Minkowski 3-space. *Int. J. Contemp. Math. Sciences*, 2(5), 221-232.
- [3] Echabbi, N., & Chahdi, A. Q. (2020). Some associated curves of normal indicatrix of a regular curve. *Journal of Mathematics Research*, 12(1), 84-97.
- [4] Turgut, A., & Erdogan, E. (1992). Involute-evolute curve couples of higher order in R^n and their horizontal lifts in R^n , *Common. Fac. Sci. Univ. Ank. Series A*, 41(3), 125-130.
- [5] Hacısalihođlu, H. H. (1983). *Diferensiyel Geometri*. Inonu University yayınları, mat., 2, 227.
- [6] Babaarslan, M., & Yaylı, Y. (2013). On helices and Bertrand curves in Euclidean 3-space. *Mathematical and Computational Applications*, 18(1), 1-11.
- [7] Izumiya, S., & Takeuchi, N. (2002). Generic properties of helices and Bertrand curves. *Journal of Geometry*, 74.
- [8] Cheng, Y. M., & Lini, C. C. (2010). On the Generalized Bertrand Curves in Euclidean-spaces. *Note di Matematica*, 29(2), 33-39.
- [9] Liu, H., & Wang, F. (2008). Mannheim partner curves in 3-space. *Journal of Geometry*, 88(1), 120-126.
- [10] Alone, M. S., Es, H., Karacan, M. K., & Bükcü, B. (2019). Mannheim curves with modified orthogonal frame in Euclidean 3-space. *Turkish Journal of Mathematics*, 43(2), 648-663.
- [11] Bishop, R. L. (1975). There is more than one way to frame a curve. *The American Mathematical Monthly*, 82(3), 246-251.
- [12] Yılmaz, S., Ünlütürk, Y., & Mađden, A. (2016). A study on the characterizations of non-null curves according to the Bishop frame of type-2 in Minkowski 3-space. *Sakarya University Journal of Science*, 20(2), 325-335.
- [13] Turhan, T. (2020). Contribution to pseudo spherical kinematics of pseudo spherical evolutes. *Malaya Journal of Matematik (MJM)*, 8(2, 2020), 622-628.
- [14] Korpınar, T., Sarıaydin, M. T., & Turhan, E. (2013). Associated curves according to Bishop frame in Euclidean 3-space. *Advanced Modeling and Optimization*, 15(3), 713-717.
- [15] Yılmaz, B. (2021). Some curve pairs according to types of Bishop frame. *AIMS Mathematics*, 6(5), 4463-4473.
- [16] Turhan, T., & Ayyıldız, N. (2016). Characterization of integral curves of a linear vector field in Lorentz 3-space. *International Journal of Geometric Methods in Modern Physics*, 13(06), 1650073.
- [17] Bukcu, B., & Karacan, M. K. (2008). On the slant helices according to Bishop frame of the timelike curve in Lorentzian space. *Tamkang Journal of Mathematics*, 39(3), 255-262.
- [18] Grbović, M., & Nešović, E. (2018). On the Bishop frames of pseudo null and null Cartan curves in Minkowski 3-space. *Journal of Mathematical Analysis and Applications*, 461(1), 219-233.
- [19] Topdal, H.T., & Turhan, T. (2022). On the Spacelike Curve with Timelike Principal Normal Pairs According to Type of Bishop Frame. *1st International Conference on Scientific and Academic Research, Konya, Türkiye*
- [20] O'Neill, B. (1983). *Semi-Riemannian geometry with applications to relativity*. Academic press.