

Increasing The Order Of Accuracy For One Nonlocal Boundary Value Problem

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Abstract

A nonlocal boundary value problem for an elliptic equation is considered. The corresponding difference problem is constructed and the error of the approximate solution is estimated with a higher order of accuracy.

Many applied problems in mechanics and physics [1], [2], [3], [4], [5] lead to nonlocal boundary value problems for partial differential equations. In this paper, we study a nonlocal boundary value problem for a quasilinear equation. The method of finite differences was applied to the numerical solution of the problem posed, and the error of the approximate solution of the nonlocal problem was estimated.

1. INTRODUCTION

Let $\Omega = \{0 < x < a, 0 < y < b\}$. Denote by $\Gamma^1 = \{0 \leq x \leq a, y = b\}$, $\Gamma^2 = \{x = 0, 0 < y < b\}$, $\Gamma^3 = \{0 \leq x \leq a, y = 0\}$, $\Gamma^4 = \{x = a, 0 < y < b\}$, $\Gamma^l = \{x = l, 0 < y < b, 0 < l < b\}$, $\Gamma = \bigcup_{i=1}^4 \Gamma^i$, $\sigma = \Gamma^1 \cup \Gamma^3$, $\bar{\Omega} = \Omega \cup \Gamma$.

Let's pretend that $f(x, y, z, p, q)$ is a given continuous function determined $\forall (x, y) \in \bar{\Omega}$ and for all z, p, q . We'll assume that the partial derivatives of f'_z, f'_p, f'_q exists and satisfies

$$f'_z \geq 0 \quad (1)$$

$$|f'_p|, |f'_q| \leq M < \infty \quad (2)$$

Let $L[u] \equiv \Delta u - f(x, y, u, u_x, u_y)$. Assume that φ, ψ are the given continuous functions of their domain definitions.

We need to find a function $u(x, y)$ continuous in $\bar{\Omega}$, twice continuously differentiable in Ω , satisfying the equation

$$L[u] = 0 \quad (3)$$

and the boundary conditions

$$u|_{\sigma} = \varphi, \quad (4)$$

$$l[u] = u(l, y) - \alpha(y)u(a, y) = \psi(y), \quad 0 < y < b, \quad (5)$$

$$\alpha(y) \geq 1, \quad 0 < y < b, \quad (6)$$

$$l^{(1)}[u] = \left(\frac{\partial u}{\partial x} + \beta(y) \frac{\partial u}{\partial y} + \delta(y)u \right) \Big|_{\Gamma^2} = \gamma(y), \quad \delta(y) \leq 0. \quad (7)$$

Let $h_1 = a/N_1$, $h_2 = b/N_2$. We construct a grid area with lines $x = x_i$, $y = y_j$, $i = 0, 1, \dots, N_1$, $j = 0, 1, \dots, N_2$ and let $x_k < l \leq x_{k+1}$.

We introduce the denotation

$$\Omega_h = \{(x_i, y_j) : i = 1, 2, \dots, N_1 - 1, j = 1, 2, \dots, N_2 - 1\},$$

$$\Gamma_h^1 = \{(x_i, b) : i = 1, 2, \dots, N_1\}, \Gamma_h^2 = \{(0, y_j) : j = 1, 2, \dots, N_2 - 1\},$$

$$\Gamma_h^3 = \{(x_i, 0) : i = 1, 2, \dots, N_1\}, \Gamma_h^4 = \{(a, y_j) : j = 1, 2, \dots, N_2 - 1\},$$

$$\sigma_h = \Gamma_h^1 \cup \Gamma_h^3, \quad \Gamma_h = \bigcup_{i=1}^4 \Gamma_h^i, \quad \overline{\Omega}_h = \Omega_h \cup \Gamma_h.$$

We approximate the operators L and l difference operators L_h, l_h defined as follows:

$$L_h[u_{ij}] \equiv \Delta_h[u_{ij}] - f(x_i, y_j, u_{ij}, D_{h_1^x}[u_{ij}], D_{h_2^y}[u_{ij}]), \quad (8)$$

$$l_h[u_{N_1j}] \equiv \frac{l - x_k}{h_1} u_{k+1j} + \frac{x_{k+1} - l}{h_1} u_k - \alpha_j u_{N_1j}, \quad (9)$$

where

$$\left. \begin{aligned} \Delta_h[u_{ij}] &= u_{\bar{x}\bar{x}} + u_{\bar{y}\bar{y}}, \quad u_{\bar{x}\bar{x}} = \frac{u_{i+1j} - 2u_{ij} + u_{i-1j}}{h_1^2}, \\ u_{\bar{y}\bar{y}} &= \frac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{h_2^2}, \quad D_{h_1^x}[u_{ij}] = \frac{u_{i+1j} - u_{i-1j}}{2h_1}, \\ D_{h_2^y}[u_{ij}] &= \frac{u_{ij+1} - u_{ij-1}}{2h_2} \end{aligned} \right\} \quad (10)$$

We formulate a difference problem corresponding to the stated problem to find a function U that is defined in $\overline{\Omega}_h$ such that

$$L_h[U_{ij}] = 0 \quad \text{in } \Omega_h, \quad (11)$$

$$l_h[U_{N_1j}] = \psi_j \quad \text{in } \Gamma_h^4, \quad (12)$$

$$U_{ij} = \varphi_{ij} \quad \text{in } \sigma_h, \quad (13)$$

$$l_h^{(1)}[U_{0j}] = \frac{U_{1j} - U_{0j}}{h_1} + \beta_j^+ \frac{U_{0j+1} - U_{0j}}{h_2} + \beta_j^- \frac{U_{0j} - U_{0j-1}}{h_2} + \delta_j U_{0j} = \gamma_j \quad \text{in } \Gamma_h^2, \quad (14)$$

where

$$\beta_j^+ = \frac{\beta_j + |\beta_j|}{2} \geq 0, \quad \beta_j^- = \frac{\beta_j - |\beta_j|}{2} \leq 0.$$

We'll assume that the domain $\overline{\Omega}_h$ is connected and the satisfies inequality

$$Mh < 2\theta, \quad (15)$$

where $h = \max\{h_1, h_2\}$, $0 < \theta < 1$ – a some fixed number.

Due to the standard scheme the following lemma is proved.

Lemma 1. Let $V \neq const$ be a function defined in $\overline{\Omega}_h$, and satisfying $\Lambda_h[V] \geq 0$ ($\Lambda_h[V] \leq 0$). Then V it may take the greatest positive (least negative) value only at the nodal points of the σ_h .

Let U be an approximate solution of the problem (11) – (14).

Theorem 1. Let the current solution u of (3) – (7) has limited third derivatives in Ω and second derivatives are continuous in $\overline{\Omega}$. Then the error $\varepsilon_{ij} = u_{ij} - U_{ij}$ of the approximate solution satisfies the equation

$$\varepsilon_{ij} = O(h).$$

Below we show that by imposing additional conditions on the function $\beta(y), \delta(y)$ the order of accuracy with in h^2 can be improved.

As can be seen from the above, it is sufficient to increase the order of approximation of the operator $l_h^{(1)}$

Assume, that $h_1 = wh^2$ ($0 < w \leq 1$) and $\beta(y), \delta(y)$ satisfy one of the following conditions

$$|\beta(y)| < w, \quad (16)$$

$$|\beta(y)| \geq w, \delta'(y) \leq 0, \quad (17)$$

$$|\beta(y)| \leq -w, \delta'(y) \geq 0. \quad (18)$$

Consider the operators

$$l_{1h}^{(1)}[U_{0j}] \equiv \frac{U_{1j} - U_{0j}}{h_1} + \beta_j \frac{U_{0j+1}U_{0j-1}}{2h_2} + \delta_j U_{0j}, \quad (19)$$

$$l_{2h}^{(1)}[U_{0j}] \equiv \frac{U_{1j} - U_{0j}}{h_1} + \beta_j \frac{U_{0j+1}U_{0j}}{h_2} + \delta_j U_{0j}, \quad (20)$$

$$l_{3h}^{(1)}[U_{0j}] \equiv \frac{U_{1j} - U_{0j}}{h_1} + \beta_j \frac{U_{0j} - U_{0j-1}}{h_2} + \delta_j U_{0j}. \quad (21)$$

Let

$$\left| \frac{\partial^p u_{0j}}{\partial x^p} \right|_{(0,j)}, \left| \frac{\partial^p u_{0j}}{\partial y^p} \right|_{(0,j)} \leq M_j^{(p)}, \quad (p \geq 1).$$

Taking into account (3), (7), (19) and applying the Taylor formula is easy to see that

$$\left| \tilde{l}_{1h}^{(1)} u_{0j} - (l^{(1)} u)_{(0,j)} \right| \leq c^{(1)} h_2^2, \quad (22)$$

where

$$\tilde{l}_{1h}^{(1)} u_{0j} \equiv l_{1h}^{(1)} u_{0j} + \frac{h_1}{2} \frac{u_{0j+1} - 2u_{0j} + u_{0j-1}}{h_2^2} - \frac{h_1}{2} f \left\{ 0, y_j, u_{0j}, D_{h_1 x}[u_{0j}], D_{h_2 y}[u_{0j}] \right\},$$

$$D_{h_1 x}[u_{ij}] = \frac{u_{i+1j} - u_{ij}}{h_1}, \quad D_{h_2 y}[u_{ij}] = \frac{u_{ij+1} - u_{ij}}{h_2},$$

$$C^{(1)} = \max_j \left\{ \frac{2(w^2 + w + \beta) + h_1 M}{12} M_j^{(3)} + \frac{w^2 M}{4} M_j^{(2)} \right\}.$$

Indeed, from (19) we have:

$$l_{1h}^{(1)}u_{0j} = (l^{(1)}u)_{(0,j)} + \frac{h_1}{2} \frac{\partial^2 u}{\partial x^2} \Big|_{(0,j)} + R_j^{(1)},$$

$$R_j^{(1)} = \frac{h_1^2}{6} \frac{\partial^3 u}{\partial x^3} \Big|_{(\xi_0^{(1)},j)} + \frac{h_2^2}{12} \left[\frac{\partial^3 u}{\partial y^3} \Big|_{(0,\eta_j^{(1)})} + \frac{\partial^3 u}{\partial y^3} \Big|_{(0,\eta_j^{(2)})} \right] \beta_j.$$

From (3) we have:

$$\frac{\partial^2 u}{\partial x^2} \Big|_{(0,j)} = -\frac{u_{0j+1} - 2u_{0j} + u_{0j-1}}{h_2^2} + f\left(0, y_j, u_{0j}, D_{h_x}[u_{0j}], D_{h_y}[u_{0j}]\right) -$$

$$-\frac{h_2}{6} \left[\frac{\partial^3 u}{\partial y^3} \Big|_{(0,\eta_j^{(3)})} - \frac{\partial^3 u}{\partial y^3} \Big|_{(0,\eta_j^{(4)})} \right] + f'_p(0, y_j, u_{0j}, p_j, q_j) \frac{\partial^2 u}{\partial x^2} \Big|_{(\xi_0^{(2)},j)} \frac{h_1}{2} +$$

$$+ f'_q(0, y_j, u_{0j}, p_j, q_j) \left[\frac{\partial^3 u}{\partial y^3} \Big|_{(0,\eta_j^{(3)})} + \frac{\partial^3 u}{\partial y^3} \Big|_{(0,\eta_j^{(4)})} \right] \frac{h_2^2}{12}.$$

Taking into account this $l_{1h}^{(1)}u_{ij}$, we get:

$$l_{1h}^{(1)}u_{0j} = (l^{(1)}u)_{(0,j)} - \frac{h_1}{2} \frac{u_{0j+1} - 2u_{0j} + u_{0j-1}}{h_2^2} +$$

$$+ \frac{h_1}{2} f'_p(0, y_j, u_{0j}, D_{h_x}[u_{0j}], D_{h_y}[u_{0j}]) + \tilde{R}_j^{(1)},$$

where

$$\tilde{R}_j^{(1)} = \frac{h_1^2}{6} \frac{\partial^3 u}{\partial x^3} \Big|_{(\xi_1^{(1)},j)} + \frac{h_2^2}{12} \left[\frac{\partial^3 u}{\partial y^3} \Big|_{(0,\eta_j^{(1)})} + \frac{\partial^3 u}{\partial y^3} \Big|_{(0,\eta_j^{(2)})} \right] \beta_j - \frac{h_1 h_2}{12} \left[\frac{\partial^3 u}{\partial y^3} \Big|_{(0,\eta_j^{(3)})} - \frac{\partial^3 u}{\partial y^3} \Big|_{(0,\eta_j^{(4)})} \right] +$$

$$+ \frac{h_1^2}{4} f'_p(0, y_j, u_{0j}, p_j, q_j) \frac{\partial^2 u}{\partial x^2} \Big|_{(\xi_0^{(2)},j)} + \frac{h_1 h_2^2}{24} f'_q(0, y_j, u_{0j}, p_j, q_j) \left[\frac{\partial^3 u}{\partial y^3} \Big|_{(0,\eta_j^{(3)})} + \frac{\partial^3 u}{\partial y^3} \Big|_{(0,\eta_j^{(4)})} \right].$$

Hence we find that

$$\tilde{l}_{1h}^{(1)}u_{0j} = (l^{(1)}u)_{(0,j)} + \tilde{R}_j^{(1)},$$

consequently,

$$\left| \tilde{l}_{1h}^{(1)}u_{0j} - (l^{(1)}u)_{(0,j)} \right| \leq \left| \tilde{R}_j^{(1)} \right|.$$

And this implies (36).

Now we prove that

$$\left| \tilde{l}_{2h}^{(1)}u_{0j} - (l^{(1)}u)_{(0,j)} \right| \leq C^{(2)} h_2^2, \quad (23)$$

where

$$\tilde{l}_{2h}^{(1)}u_{0j} \equiv l_{2h}^{(1)}u_{0j} + \frac{\beta_j h_2 - h_1}{2\beta_j} D_{h_1 h_2 xy}[u_{0j}] + \frac{\delta_j}{\beta_j} (\beta_j h_2 - h_1) D_{h_2 y}[u_{0j}] + \frac{\delta'_j}{2\beta_j} (\beta_j h_2 - h_1) u_{0j} -$$

$$- \frac{\gamma'_j}{2\beta_j} (\beta_j h_2 - h_1) - \frac{h_1}{2} f(0, y_j, u_{0j}, D_{h_x}[u_{0j}], D_{h_y}[u_{0j}]),$$

$$C^{(2)} = \max_j \left\{ \left[\frac{\beta_j + w^2}{6} + \frac{(\beta_j - w)(1-w)}{4\beta_j} + \frac{h_1 M}{12} \right] M_i^{(3)} + \left[\frac{|\delta_j + \beta'_j|(\beta_j - w)}{4\beta_j} + \frac{w^2 M}{4} \right] M_j^{(2)} \right\},$$

$$D_{h_1 h_2 xy}[u_{0j}] = D_{h_1 x} \{ D_{h_2 y}[u_{0j}] \}.$$

Suppose that $\beta(y) \neq 0$. Then from (7) we have:

$$\frac{\partial^2 u(0, u)}{\partial y^2} = -\frac{1}{\beta(y)} \frac{\partial^2 u(0, y)}{\partial x \partial y} - \frac{\delta'(y)}{\beta(y)} u(0, y) - \frac{\delta(y) + \beta'_j}{\beta(y)} \frac{\partial u(0, y)}{\partial y} + \frac{\gamma'(y)}{\beta(y)}. \quad (24)$$

Obviously

$$l_{2h}^{(1)} u_{0j} = (l_3 u)_{(0,j)} + \frac{h_1}{2} \frac{\partial^2 u}{\partial x^2} \Big|_{(0,j)} + \frac{h_2}{2} \beta_j \frac{\partial^2 u}{\partial y^2} \Big|_{(0,j)} + R_j^{(2)},$$

where

$$R_j^{(2)} = \frac{h_1^2}{6} \frac{\partial^3 u}{\partial x^3} \Big|_{(\xi_0^{(1)}, j)} + \beta_j \frac{h_2^2}{6} \frac{\partial^3 u}{\partial y^3} \Big|_{(0, \eta_j^{(1)})}.$$

From (3) we get:

$$\frac{\partial^2 u}{\partial x^2} \Big|_{(0,j)} = -\frac{\partial^2 u}{\partial y^2} \Big|_{(0,j)} + f \left(0, y_j, u_{0j}, \frac{\partial u}{\partial x} \Big|_{(0,j)}, \frac{\partial u}{\partial y} \Big|_{(0,j)} \right).$$

Then

$$l_{2h}^{(1)} u_{0j} = (l^{(1)} u)_{(0,j)} + \frac{1}{2} (\beta_j h_2 - h_1) \frac{\partial^2 u}{\partial y^2} \Big|_{(0,j)} + \frac{h_1}{2} f \left(0, y_j, u_{0j}, \frac{\partial u}{\partial x} \Big|_{(0,j)}, \frac{\partial u}{\partial y} \Big|_{(0,j)} \right) + R_j^{(2)}.$$

Taking into account (24)

$$\begin{aligned} l_{2h}^{(1)} u_{0j} &= (l^{(1)} u)_{(0,j)} + \frac{1}{2} (\beta_j h_2 - h_1) \left[-\frac{1}{\beta_j} \frac{\partial^2 u}{\partial x \partial y} \Big|_{(0,j)} - \frac{\delta'_j}{\beta_j} u_{0j} - \frac{\delta_j + \beta'_j}{\beta_j} \frac{\partial u}{\partial y} \Big|_{(0,j)} + \frac{\gamma_j}{\beta_j} \right] + \\ &+ \frac{h_1}{2} f \left(0, y_j, u_{0j}, \frac{\partial u}{\partial x} \Big|_{(0,j)}, \frac{\partial u}{\partial y} \Big|_{(0,j)} \right) + R_j^{(2)} = (l^{(1)} u)_{(0,j)} - \frac{1}{2\beta_j} (\beta_j h_2 - h_1) \frac{\partial^2 u}{\partial x \partial y} \Big|_{(0,j)} - \\ &- \frac{\delta_j + \beta'_j}{2\beta_j} (\beta_j h_2 - h_1) \frac{\partial u}{\partial y} \Big|_{(0,j)} - \frac{\delta'_j}{2\beta_j} (\beta_j h_2 - h_1) u_{0j} + \frac{\gamma'_j}{2\beta_j} (\beta_j h_2 - h_1) + \\ &+ \frac{h_1}{2} f \left(0, y_j, u_{0j}, \frac{\partial u}{\partial x} \Big|_{(0,j)}, \frac{\partial u}{\partial y} \Big|_{(0,j)} \right) + R_j^{(2)} = (l^{(1)} u)_{(0,j)} - \frac{\beta_j h_2 - h_1}{2\beta_j} D_{h_1 h_2 xy}[u_{0j}] - \\ &- \frac{\delta_j + \beta'_j}{\beta_j} (\beta_j h_2 - h_1) D_{h_2 y}[u_{0j}] - \frac{\delta'_j}{2\beta_j} (\beta_j h_2 - h_1) u_{0j} + \frac{\gamma'_j}{2\beta_j} (\beta_j h_2 - h_1) + \\ &+ \frac{h_1}{2} f \left(0, y_j, u_{0j}, D_{h_1 x}[u_{0j}], D_{h_2 y}[u_{0j}] \right) + \tilde{R}_j^{(2)}, \end{aligned}$$

where

$$\begin{aligned} \tilde{R}_j^{(2)} = & R_j^{(2)} - \frac{\beta_j h_2 - h_1}{4\beta_j} \left[\frac{\partial^3 u}{\partial x^2 \partial y} h_1 - \frac{\partial^3 u}{\partial x \partial y^2} h_2 \right] - \\ & - \frac{\delta_j + \beta'_j}{4\beta_j} (\beta_j h_2 - h_1) h_2 \frac{\partial^2 u}{\partial y^2} \Big|_{(0, \eta_j^{(2)})} + \frac{h_1^2}{4} f'_p(0, y_j, u_{0j}, p_j, q_j) \frac{\partial^2 u}{\partial x^2} \Big|_{(\xi_0^{(2)}, j)} + \\ & + \frac{h_1 h_2^2}{24} f'_q(0, y_j, u_{0j}, p_j, q_j) \left[\frac{\partial^3 u}{\partial y^3} \Big|_{(0, \eta_j^{(2)})} + \frac{\partial^3 u}{\partial y^3} \Big|_{(0, \eta_j^{(3)})} \right]. \end{aligned}$$

Then

$$\left| \tilde{l}_{2h}^{(1)} u_{0j} - (l_3 u)_{(0,j)} \right| \leq \left| \tilde{R}_j^{(2)} \right|.$$

This implies (23).

Finally, we prove that

$$\left| \tilde{l}_{3h}^{(1)} u_{0j} - (l_3 u)_{(0,j)} \right| \leq C^{(2)} h_2^2, \quad (25)$$

where

$$C^{(3)} = \left(\frac{|w^2 + \beta|}{6} + \frac{|w + \beta|(w + 1)}{2|\beta|} + \frac{h_1 M}{12} \right) M_3 + \left(\frac{|w + \beta| |\delta + \beta'|}{4|\beta|} + \frac{w^2 M}{4} \right) M_2,$$

$$\begin{aligned} \tilde{l}_{3h}^{(1)} u_{0j} = & l_{3h}^{(1)} u_{0j} - \frac{\beta_j h_2 - h_1}{2\beta_j} D_{h_1 h_2 x y} [u_{0j}] - \frac{\delta_j}{\beta_j} (\beta_j h_2 + h_1) D_{h_2 y} [u_{0j}] - \frac{\delta'_j}{2\beta_j} (\beta_j h_2 + h_1) u_{0j} + \\ & + \frac{\gamma'_j}{2\beta_j} (\beta_j h_2 - h_1) + \frac{h_1}{2} f(0, y_j, u_{0j}, D_{h_1 x} [u_{0j}], D_{h_2 y} [u_{0j}]), \\ & D_{h_1 h_2 x y} [u_{0j}] = D_{h_1 x} \{ D_{h_2 y} [u_{0j}] \} \end{aligned}$$

Indeed,

$$\begin{aligned} l_{3h}^{(1)} u_{0j} = & (l^{(1)} u)_{(0,j)} - \frac{h_1 + h_2 \beta_j}{2} \frac{\partial^2 u}{\partial y^2} \Big|_{(0,j)} - \frac{h_1}{2} f \left(0, y_j, u_{0j}, \frac{\partial u}{\partial x} \Big|_{(0,j)}, \frac{\partial u}{\partial y} \Big|_{(0,j)} \right) + R_j^{(3)}, \\ R_j^{(3)} = & \frac{h_1^2}{6} \frac{\partial^3 u}{\partial x^3} \Big|_{(\xi_0^{(1)}, j)} + \beta_j \frac{h_2^2}{6} \frac{\partial^3 u}{\partial y^3} \Big|_{(0, \eta_j^{(1)})}. \end{aligned}$$

Taking into account (24)

$$\begin{aligned} l_{3h}^{(1)} u_{0j} = & (l^{(1)} u)_{(0,j)} + \frac{h_1 + h_2 \beta_j}{2\beta_j} \frac{\partial^2 u}{\partial x \partial y} + \frac{h_1 + h_2 \beta_j}{2} \frac{\delta'_j}{\beta_j} u_{0j} + \frac{h_1 + h_2 \beta_j}{2} \frac{\delta_j + \beta'_j}{\beta_j} \frac{\partial u}{\partial y} \Big|_{(0,j)} - \\ & - \frac{h_1 + h_2 \beta_j}{2} \frac{\gamma'_j}{\beta_j} - \frac{h_1}{2} f \left(0, y_j, u_{0j}, \frac{\partial u}{\partial x} \Big|_{(0,j)}, \frac{\partial u}{\partial y} \Big|_{(0,j)} \right) + R_j^{(3)}, \\ l_{3h}^{(1)} u_{0j} = & (l^{(1)} u)_{(0,j)} + \frac{h_1 + h_2 \beta_j}{2\beta_j} D_{h_1 h_2 x y} [u_{0j}] + \frac{(h_1 + h_2 \beta_j)(\delta_j + \beta')}{2\beta_j} D_{h_2 y} [u_{0j}] + \\ & + \frac{(h_1 + h_2 \beta_j)\delta'_j}{2\beta_j} u_{0j} - \frac{(h_1 + h_2 \beta_j)\gamma'_j}{2\beta_j} - \frac{h_1}{2} f(0, y_j, u_{0j}, D_{h_1 x} [u_{0j}], D_{h_2 y} [u_{0j}]) + \tilde{R}_j^{(3)}, \end{aligned}$$

where

$$\begin{aligned} \tilde{R}_j^{(3)} = & R_j^{(3)} + \frac{h_1 + h_2\beta_j}{2\beta_j} \left[\frac{\partial^3 u}{\partial x^2 \partial y} h_1 + \frac{\partial^3 u}{\partial x \partial y^2} h_2 \right] + \\ & + \frac{(h_1 + h_2\beta_j)(\delta_j + \beta'_j)}{4\beta_j} h^2 \frac{\partial^2 u}{\partial y^2} \Big|_{(0, \eta_j^{(2)})} - \frac{h_1^2}{4} f'_p(0, y_j, u_{0j}, p_j, q_j) \frac{\partial^2 u}{\partial x^2} \Big|_{(\xi_0^{(2)}, j)} - \\ & - \frac{h_1 h_2^2}{24} f'_q(0, y_j, u_{0j}, p_j, q_j) \left[\frac{\partial^3 u}{\partial y^3} \Big|_{(0, \eta_j^{(3)})} + \frac{\partial^3 u}{\partial y^3} \Big|_{(0, \eta_j^{(4)})} \right]. \end{aligned}$$

Consequently,

$$\left| \tilde{l}_{3h}^{(1)} u_{0j} - (l^{(1)} u)_{(0,j)} \right| \leq \left| \tilde{R}_j^{(3)} \right|,$$

Which was required to prove.

We now state the difference problem corresponding to the problem (3) - (7).

It is required to find a discrete function $U_{ij}^{(k)}$ ($k = 1, 2, 3$) determined in $\bar{\Omega}_h$ satisfying the properties

(11) - (13), and one of the following conditions

$$\tilde{l}_{kh}^{(1)} U_{0j} = \gamma_j \quad (j = \overline{1, N_2 - 1}, \quad k = 1, 2, 3) \quad (26)$$

respectively, when one of the conditions (16), (17) and (18) is satisfied. The solution of the difference scheme (11) - (13), with one of the conditions (26) will be taken as an approximate solution of the problem (3) - (7) at the points $\bar{\Omega}_h$.

Consider the following linear difference operators:

$$\Lambda_h^{(k)} [U_{ij}] = \begin{cases} \tilde{L}_h [U_{ij}], \\ l_h [U_{N_1 j}], \\ \bar{l}_{kh}^{(1)} [U_{0j}], \quad (k = 1, 2, 3), \end{cases}$$

where

$$\tilde{L}_h [U_{ij}] \equiv \Delta_h [U_{ij}] + \xi_{ij} D_{h_1 x} [U_{ij}] + \eta_{ij} D_{h_2 y} [U_{ij}] - \mu_{ij} U_{ij},$$

$$\bar{l}_{1h}^{(1)} [U_{0j}] \equiv l_{1h}^{(1)} [U_{0j}] + \frac{h_1}{2} \frac{U_{0j+1} - 2U_{0j} + U_{0j-1}}{h_2^2} -$$

$$- \frac{h_1}{2} \left[\xi_{0j} D_{h_1 x} [U_{0j}] + \eta_{0j} D_{h_2 y} [U_{0j}] - \mu_{0j} U_{0j} \right],$$

$$\bar{l}_{2h}^{(1)} [U_{0j}] \equiv l_{2h}^{(1)} [U_{0j}] + \frac{\beta_j h_2 - h_1}{2\beta_j} D_{h_1 h_2 xy} [U_{0j}] +$$

$$+ \frac{\delta_j}{\beta_j} (\beta_j h_2 - h_1) D_{h_2 y} [U_{0j}] + \frac{\delta'_j}{2\beta_j} (\beta_j h_2 - h_1) U_{0j} -$$

$$- \frac{h_1}{2} \left[\xi_{0j} D_{h_1 x} [U_{0j}] + \eta_{0j} D_{h_2 y} [U_{0j}] - \mu_{0j} U_{0j} \right]$$

$$\begin{aligned} \bar{l}_{3h}^{(1)}[U_{0j}] &\equiv l_{3h}^{(1)}[U_{0j}] - \frac{\beta_j h_2 - h_1}{2\beta_j} D_{h_1 h_2 xy}[U_{0j}] - \\ &- \frac{\delta_j}{\beta_j} (\beta_j h_2 + h_1) D_{h_2 y}[U_{0j}] + \frac{\delta'_j}{2\beta_j} (\beta_j h_2 + h_1) U_{0j} + \\ &+ \frac{h_1}{2} \left[\xi_{0j} D_{h_1 x}[U_{0j}] + \eta_{0j} D_{h_2 y}[U_{0j}] - \mu_{0j} U_{0j} \right] \end{aligned}$$

We assume that if (16) is satisfied, then

$$Mh_2 < 2(1 - \sup|\beta(x)|), \quad (27)$$

and if the (17), (18) are satisfied, then

$$\bar{M}h_2 < 1, \quad (28)$$

where

$$\bar{M} = \max \left\{ \frac{M}{1 + (\sup|\beta|)^{-1}}, \frac{M + \sup \left(\frac{|\beta| + 1}{|\beta|} |\beta' + \delta| \right)}{\inf |\beta| + (\sup|\beta|)^{-1}} \right\}.$$

Lemma 2. Let $V \neq const$ be a function defined in $\bar{\Omega}_h$, that satisfies the inequality $\Lambda_h^{(k)}[V_{ij}] \geq 0$ ($\Lambda_h^{(k)}[V_{ij}] \leq 0$) $k = 1, 2, 3$. Then V may take the greatest positive (least negative) value only at the points σ_h .

Proof. It's obvious that

$$\bar{l}_{1h}^{(1)}[U_{ij}] \equiv A_{1j}^{(1)}U_{1j} + A_{2j}^{(1)}U_{0j-1} + A_{3j}^{(1)}U_{0j+1} - A_{0j}^{(1)}U_{0j},$$

$$\bar{l}_{2h}^{(1)}[U_{ij}] \equiv A_{1j}^{(2)}U_{1j} + A_{2j}^{(2)}U_{1j+1} + A_{3j}^{(2)}U_{0j+1} - A_{0j}^{(2)}U_{0j},$$

$$\bar{l}_{3h}^{(1)}[U_{ij}] \equiv A_{1j}^{(3)}U_{1j} + A_{2j}^{(3)}U_{0j-1} + A_{3j}^{(3)}U_{0j-1} - A_{0j}^{(3)}U_{0j},$$

where

$$A_{0j}^{(1)} = \frac{h_1}{h_2^2} + \frac{1}{h_1} - \delta_j + \frac{\xi_j}{2} + \frac{h_1}{2} \mu_j, \quad A_{1j}^{(1)} = \frac{1}{h_1} \left(1 - \frac{h_1}{2} \xi_j \right),$$

$$A_{2j}^{(1)} = \frac{1}{2h_2} \left(\frac{h_1}{h_2} - \beta_j - \frac{h_1}{2} \eta_j \right), \quad A_{3j}^{(1)} = \frac{1}{2h_2} \left(\frac{h_1}{h_2} + \beta_j + \frac{h_1}{2} \eta_j \right),$$

$$A_{0j}^{(2)} = \beta_j \left(\frac{1}{h_1} + \frac{1}{h_2} \right) - \delta_j - \frac{\beta_j h_2 - h_1}{2\beta_j h_2 h_1} + \frac{\delta_j}{h_2 \beta_j} (\beta_j h_2 - h_1) -$$

$$- \frac{\delta'_j}{2\beta_j} (\beta_j h_2 - h_1) - \frac{\xi_j}{2} - \frac{h_1}{2h_2} \eta_j - \frac{h_1}{2} \mu_j,$$

$$A_{1j}^{(2)} = \frac{\beta_j}{h_1} - \frac{\beta_j h_2 - h_1}{2\beta_j h_2 h_1} - \frac{\xi_j}{2}, \quad A_{2j}^{(2)} = \frac{\beta_j h_2 - h_1}{2\beta_j h_2 h_1},$$

$$\begin{aligned}
A_{3j}^{(2)} &= \frac{\beta_j}{h_2} - \frac{\beta_j h_2 - h_1}{2\beta_j h_2 h_1} + \frac{\delta_j}{h_2 \beta_j} (\beta_j h_2 - h_1) - \frac{h_1}{2h_2} \eta_j, \\
A_{0j}^{(3)} &= \frac{1}{h_1} - \frac{\beta_j}{h_2} - \delta_j - \frac{\beta_j h_2 - h_1}{2\beta_j h_1 h_2} + \frac{\delta_j (\beta_j h_2 + h_1)}{h_2 \beta_j} + \\
&+ \frac{\delta'_j}{2\beta_j} (\beta_j h_2 + h_1) + \frac{\xi_j}{2} - \frac{h_1}{2h_2} \eta_j + \frac{h_1}{2} \mu_j, \\
A_{1j}^{(3)} &= \frac{1}{h_1} - \frac{\beta_j h_2 - h_1}{2\beta_j h_1 h_2} + \frac{\xi_j}{2}, \\
A_{2j}^{(3)} &= -\frac{\beta_j}{h_2} - \frac{\beta_j h_2 - h_1}{2\beta_j h_1 h_2} + \frac{\delta_j}{h_2 \beta_j} (\beta_j h_2 + h_1) - \frac{h_1}{2h_2} \eta_j, \\
A_{3j}^{(3)} &= \frac{\beta_j h_2 - h_1}{2\beta_j h_1 h_2}.
\end{aligned}$$

All these coefficients are positive and satisfy the following conditions:

$$\begin{aligned}
A_{0j}^{(1)} - A_{1j}^{(1)} - A_{2j}^{(1)} - A_{3j}^{(1)} &= -\delta_j + \frac{h_1}{2} \mu_j \geq 0, \\
A_{0j}^{(2)} - A_{1j}^{(2)} - A_{2j}^{(2)} - A_{3j}^{(2)} &= -\delta_j - \frac{h_1}{2} \mu_j - \frac{\delta'_j}{2\beta_j} (\beta_j h_2 - h_1) \geq 0, \\
A_{0j}^{(3)} - A_{1j}^{(3)} - A_{2j}^{(3)} - A_{3j}^{(3)} &= -\delta_j + \frac{\delta'_j}{2\beta_j} (\beta_j h_2 + h_1) + \frac{h_1}{2} \mu_j \geq 0.
\end{aligned}$$

Taking into account these properties of the coefficients, applying Lemma 1 we obtain Lemma 2.

Corollary. Lemma 2 implies that the solution of (11) - (13) (26) is unique.

Theorem 2. Let u the exact solution of the problem (3) - (7) limited the fourth derivatives and continued in the third derivative $\bar{\Omega}$. Then the error $\varepsilon_{ij} = u_{ij} - U_{ij}$, where U_{ij} - the approximate solution of (11)-(13), (26), the estimate $\varepsilon = O(h^2)$.

Proof. With the help of Taylor's formula for the error $\varepsilon_{ij} = u_{ij} - U_{ij}$ we have:

$$\begin{cases}
\tilde{L}_h[\varepsilon_{ij}] = O(h^2) & \text{in } \Omega_h, \\
l_h[\varepsilon_{N_{1j}}] = O(h^2) & \text{in } \Gamma_h^4, \\
\varepsilon_{ij} = 0 & \text{in } \sigma_h, \\
\bar{l}_{kh}^{(1)}[\varepsilon_{0j}] = O(h^2), k = 1, 2, 3 & \text{in } \Gamma_h^2.
\end{cases} \quad (29)$$

As in the proof of Theorem 1, we represent the solution of the system (29) of the form

$$\varepsilon_{ij} = \varepsilon_{ij}^1 + \varepsilon_{ij}^2,$$

where

$$\left. \begin{aligned} \tilde{L}_h[\varepsilon_{ij}^1] &= O(h^2) \quad \text{in } \Omega_h, \\ \varepsilon_{N_{1j}}^1 &= 0 \quad \text{in } \Gamma_h^4, \\ \varepsilon_{ij}^1 &= 0 \quad \text{in } \sigma_h, \\ \bar{l}_{kh}^{(1)}[\varepsilon_{ij}^1] &= O(h^2), k = 1,2,3 \quad \text{in } \Gamma_h^2, \end{aligned} \right\} \quad (30)$$

$$\left. \begin{aligned} \tilde{L}_h[\varepsilon_{ij}^2] &= 0 \quad \text{in } \Omega_h, \\ l_h[\varepsilon_{N_{1j}}^2] &= -l_h[\varepsilon_{N_{1j}}^1] + O(h^2) \quad \text{in } \Gamma_h^4, \\ \varepsilon_{ij}^2 &= 0 \quad \text{in } \sigma_h, \\ \bar{l}_{kh}^{(1)}[\varepsilon_{0j}^2] &= 0, k = 1,2,3 \quad \text{in } \Gamma_h^2. \end{aligned} \right\} \quad (31)$$

An estimate of $\max_{\Omega_h} |\varepsilon_h^1| \leq c_7 h^2$ for the solutions system of (30) is obtained on the basis of Lemma 2, by the majorant function

$$g(x, y) = \frac{1}{k} (e^{v_0 x} - e^{v_0 y}),$$

and the parameters k and v_0 are selected as follows:

$$k = \mu_0 v_0, \quad \mu_0 = \min\{\alpha^0, M\beta^0\},$$

$$\alpha^0 = \begin{cases} \sup|\beta| & \text{if } |\beta| < 1, \\ \frac{1-\theta}{2} & \text{if } |\beta| \geq 1, \end{cases}$$

$$\beta^0 = \begin{cases} \sup|\beta| & \text{if } |\beta| < 1, \\ 1-\theta & \text{if } |\beta| \geq 1, \end{cases}$$

$$v_0 = \frac{2M}{\delta} \operatorname{arcth} \left(\frac{2\bar{\delta} - \delta^2}{2} \right),$$

$$\bar{\delta} = \begin{cases} 1 - \sup|\beta| & \text{if } |\beta| < 1, \\ \theta & \text{if } |\beta| \geq 1. \end{cases}$$

An estimate of $\max_{\Omega_h} |\varepsilon_h^2| \leq c_8 h^2$ for the solutions of the system (31).

Theorem 2 is proved.

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