Original Article

# Numerical Solution And Error Evaluation Of A Nonlocal Boundary Problem For Elliptic Equation In Partial Derivatives

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## Abstract

In a rectangular domain, a nonlocal boundary value problem for an elliptic equation is considered. The corresponding difference problem was constructed and the error of the approximate solution was estimated.

Many applied problems of heat conduction [1], [2], [3], fluid mechanics [4], and the theory of elasticity and shells [5] lead to nonlocal boundary value problems for partial differential equations. Non-local boundary conditions are especially difficult for justification of classical finite difference schemes due to the complexity of the structure of the matrices obtained from systems of equations. This difficulty manifests itself especially in the justification of numerical methods in the case of non-linear equations. In this paper, we study a nonlocal boundary value problem for a quasilinear equation. The method of finite differences was applied to the numerical solution of the problem posed, and the error of the approximate solution of the nonlocal problem was estimated.

## 1.INTRODUCTION

Let $\Omega = \{o < x < a,\ 0 < y < b\}$. Denote by $\Gamma^1 = \{0 \le x \le a,\ \ y = b\}$, $\Gamma^2 = \{x = 0,\ 0 < y < b\}$, $\Gamma^3 = \{0 \le x \le a,\ \ y = 0\}$, $\Gamma^4 = \{x = a,\ 0 < y < b\}$, $\Gamma^l = \{x = l, 0 < y < b, 0 < l < b\}$, $\Gamma = \bigcup_{i=1}^{4} \Gamma^i$, $\sigma = \Gamma^1 \bigcup \Gamma^3$, $\overline{\Omega} = \Omega \bigcup \Gamma$.

Let's pretend that $f(x, y, z, p, q)$ is a given continuous function determined $\forall (x, y) \in \overline{\Omega}$ and for all $z, p, q$. We'll assume that the partial derivatives of $f'_z, f'_p, f'_q$ exists and satisfies

$$f'_z \ge 0 \qquad (1)$$

$$| f'_p |, | f'_q | \le M < \infty \qquad (2)$$

Let $L[u] \equiv \Delta u - f(x, y, u, u_x, u_y)$. Assume that $\varphi, \psi$ are the given continuous functions of their domain definitions.

We need to find a function $u(x,y)$ continuous in $\overline{\Omega}$ ,twice continuously differentiable in $\Omega$ ,satisfying the equation

$$L[u]=0 \qquad (3)$$

and the boundary conditions

$$u\big|_{\sigma}=\varphi\,, \qquad (4)$$

$$l[u]=u(l,y)-\alpha(y)u(a,y)=\psi(y),\quad 0<y<b, \qquad (5)$$

$$\alpha(y)\geq 1,\;\; 0<y<b, \qquad (6)$$

$$l^{(1)}[u]=\left(\frac{\partial u}{\partial x}+\beta(y)\frac{\partial u}{\partial y}+\delta(y)u\right)\Bigg|_{\Gamma^2}=\gamma(y),\quad \delta(y)\leq 0\,. \qquad (7)$$

Let $h_1=a/N_1,\; h_2=b/N_2$. We construct a grid area with lines $x=x_i$, $y=y_j$, $i=0,1,...N_1,\; j=0,1,...N_2$ and let $x_k<l\leq x_{k+1}$.

We introduce the denotation

$$\Omega_h=\{(x_i,y_j): i=1,2,...,N_1-1,\;\; j=1,2,...,N_2-1\},$$

$$\Gamma_h^1=\{(x_i,b): i=1,2,...,N_1\},\; \Gamma_h^2=\{(0,y_j):\; j=1,2,...,N_2-1\},$$

$$\Gamma_h^3=\{(x_i,0): i=1,2,...,N_1\},\; \Gamma_h^4=\{(a,y_j):\; j=1,2,...,N_2-1\},$$

$$\sigma_h=\Gamma_h^1\cup\Gamma_h^3,\quad \Gamma_h=\bigcup_{i=1}^{4}\Gamma_h^i,\; \overline{\Omega}_h=\Omega_h\cup\Gamma_h\,.$$

We approximate the operators $L$ and $l$ difference operators $L_h,\,l_h$ defined as follows:

$$L_h[u_{ij}]\equiv\Delta_h[u_{ij}]-f(x_i,y_j,u_{ij},D_{h_1\overset{\circ}{x}}[u_{ij}],D_{h_2\overset{\circ}{y}}[u_{ij}]), \qquad (8)$$

$$l_h[u_{N_1 j}]\equiv\frac{l-x_k}{h_1}u_{k+1\,j}+\frac{x_{k+1}-l}{h_1}u_k-\alpha_j u_{N_1 j}, \qquad (9)$$

where

$$\left.\begin{aligned}
&\Delta_h[u_{ij}]=u_{\bar{x}x}+u_{\bar{y}y},\; u_{\bar{x}x}=\frac{u_{i+1\,j}-2u_{ij}+u_{i-1\,j}}{h_1^2},\\
&u_{\bar{y}y}=\frac{u_{ij+1}-2u_{ij}+u_{ij-1}}{h_2^2},\; D_{h_1\overset{\circ}{x}}[u_{ij}]=\frac{u_{i+1\,j}-u_{i-1\,j}}{2h_1},\\
&D_{h_2\overset{\circ}{y}}[u_{ij}]=\frac{u_{ij+1}-u_{ij-1}}{2h_2}
\end{aligned}\right\} \qquad (10)$$

Let us construct a difference problem corresponding to the stated problem to find a function $U$ that is defined in $\overline{\Omega}_h$ such that

$$L_h[U_{ij}]=0 \text{ in } \Omega_h, \qquad (11)$$

$$l_h[U_{N_1 j}]=\psi_j \text{ in } \Gamma_h^4, \qquad (12)$$

$$U_{ij}=\varphi_{ij} \text{ in } \sigma_h, \qquad (13)$$

$$l_h^{(1)}[U_{0j}]=\frac{U_{1j}-U_{0j}}{h_1}+\beta_j^{+}\frac{U_{0j+1}-U_{0j}}{h_2}+\beta_j^{-}\frac{U_{0j}-U_{0j-1}}{h_2}+\delta_j U_{0j}=\gamma_j \text{ in } \Gamma_h^2, \quad (14)$$

where

$$\beta_j^+ = \frac{\beta_j + |\beta_j|}{2} \geq 0, \qquad \beta_j^- = \frac{\beta_j - |\beta_j|}{2} \leq 0 .$$

We'll assume that the domain $\overline{\Omega}_h$ is connected and the satisfies inequality

$$Mh < 2\theta , \qquad (15)$$

where $h = \max\{h_1, h_2\}, \quad 0 < \theta < 1$ – a some fixed number.

## 2. RESULTS

Consider the linear difference operator

$$\Lambda_h[U_{ij}] = \begin{cases} \Lambda'_h[U_{ij}] \ \text{ in } \ \Omega_h, \\ l_h[U_{N_1 j}] \ \text{ in } \ \Gamma_h^4, \\ l_h^{(1)}[U_{0j}] \ \text{ in } \ \Gamma_h^2, \end{cases} \qquad (16)$$

where

$$\Lambda'_h[U_{ij}] = \Delta_h[U_{ij}] + \xi_{ij} D_{h_1 \overset{o}{x}} [U_{ij}] + \eta_{ij} D_{h_2 \overset{o}{y}} [U_{ij}] - \mu_{ij} U_{ij} ,$$

$$|\xi_{ij}|, |\eta_{ij}| \leq M , \qquad (17)$$

$$\mu_{ij} \geq 0. \qquad (18)$$

Due to the standard scheme the following lemma is proved.

**Lemma 1.** Let $V \not\equiv const$ be a function defined in $\overline{\Omega}_h$, and satisfying $\Lambda_h[V] \geq 0$ ( $\Lambda_h[V] \leq 0$ ).Then $V$ it may take the greatest positive (least negative) value only at the nodal points of the $\sigma_h$.

Let $U$ be an approximate solution of the problem (11) – (14).

**Theorem 1.** Let the current solution $u$ of (3) – (7) has limited third derivatives in $\Omega$ and second derivatives are continuous in $\overline{\Omega}$. Then the error $\varepsilon_{ij} = u_{ij} - U_{ij}$ of the approximate solution satisfies the equation

$$\varepsilon_{ij} = O(h) .$$

**Proof**. On the basis of Taylor's formula, we have

$$\begin{cases} \Lambda'_h[\varepsilon_{ij}] = O(h) \ \ in \ \Omega_h, \\ l_h[\varepsilon_{N_1 j}] = O(h^2) \ \text{ in } \ \Gamma_h^4, \\ \varepsilon_{ij} = 0, \ \text{ in } \ \sigma_h, \\ l_h^{(1)}[\varepsilon_{0j}] = O(h) \ \text{ in } \ \Gamma_h^2. \end{cases} \qquad (19)$$

We represent the solution of (19) as

$$\varepsilon_{ij} = \varepsilon_{ij}^1 + \varepsilon_{ij}^2 , \qquad (20)$$

where

$$\begin{cases} \Lambda'_h[\varepsilon_{ij}^1] = O(h) \ \text{ in } \ \Omega_h, \\ \varepsilon_{N_1 j}^1 = 0 \ \text{ in } \ \Gamma_h^4, \\ \varepsilon_{ij}^1 = 0, \ \ in \ \sigma_h, \\ l_h^{(1)}[\varepsilon_{0j}^1] = O(h) \ \text{ in } \ \Gamma_h^2. \end{cases} \qquad (21)$$

$$\begin{cases} \Lambda'_h[\varepsilon_{ij}^2] = 0 \ \ in\ \Omega_h, \\ l_h[\varepsilon_{N_1 j}^2] = -l_h[\varepsilon_{N_1 j}^1] + O(h^2) \ \text{ in } \ \Gamma_h^4, \\ \varepsilon_{ij}^2 = 0, \ \text{ in } \ \sigma_h, \\ l_h^{(1)}[\varepsilon_{0j}^2] = 0 \ \text{ in } \ \Gamma_h^2. \end{cases} \qquad (22)$$

First, we estimate the system (21). Consider the function

$$g(x,y) = \frac{1}{K}(e^{\nu_0 a} - e^{\nu_0 x}),$$

where

$$\nu_0 = \frac{M}{\theta} arcth\left(\frac{3\theta - \theta^2}{2}\right), \ \ k = \mu_0 \nu_0, \ \ \mu_0 = \min\left\{1, \frac{M}{2}(1-\theta)\right\}.$$

It is easy to verify, that

$$\begin{cases} \Lambda'_h[g_{ij}] \le -1 \ \text{ in } \ \Omega_h, \\ l_h^{(1)}[g_{0j}] \le -1 \ \text{ in } \ \Gamma_h^2. \end{cases} \qquad (23)$$

On the basis of (21), (23) and Lemma 1 we get that the function

$$G_{ij}^{\pm} = c \cdot h \cdot g_{ij} \pm \varepsilon_{ij}^1$$

Is positive on $\overline{\Omega}_h$ (for the selected finite constant $C$).

from this inequality It follows that

$$\max_{\Omega_h} |\varepsilon_{ij}^1| \le C_1 h, \ \ C_1 = const > 0. \qquad (24)$$

Denote by $w = \max_{\Gamma_h^4} |\varepsilon_{N_1 j}^2|$ and let the $\overline{\omega}_{ij}$ - be the solution of

$$\Lambda'_h[\overline{\omega}_{ij}] = 0 \ \text{ in } \Omega_h,$$

$$\overline{\omega}_{N_1 j} = w \ \text{ in } \Gamma_h^4,$$

$$\overline{\omega}_{ij} = 0 \ \text{ in } \sigma_h,$$

$$l_h^{(1)}[\overline{\omega}_{0j}] = 0 \ \text{ in } \Gamma_h^2.$$

Lemma 1 implies that

$$|\varepsilon_{ij}^2| \le \overline{\omega}_{ij} \ \text{ in } \overline{\Omega}_h, \qquad (25)$$

$$\overline{\omega}_{ij} \le \tau_i w, \ \ 0 < \tau_i < 1 \text{ in } \Omega_h. \qquad (26)$$

On the other hand

$$l_h[\varepsilon_{N_1 j}^2] = -l_h[\varepsilon_{N_1 j}^1] + O(h^2) \ \text{ in } \Gamma_h^4$$

Hence, respectively to (25), (26) we have

$$\alpha_j |\varepsilon_{N_1 j}^2| \le \frac{l - x_k}{h_1}|\varepsilon_{k+1 j}^2| + \frac{x_{k+1} - l}{h_1}|\varepsilon_{kj}^2| + \frac{l - x_k}{h_1}|\varepsilon_{k+1 j}^1| + \frac{x_{k+1} - l}{h_1}|\varepsilon_{kj}^1| + C_2 h^2$$

or

$$\alpha_j w \le \tau w + C_1 h + C_2 h,$$

where

$$\tau = \max\{\tau_{k+1}, \tau_k\}.$$

Hence we have

$$w \le \frac{C_3 h}{\alpha_j - \kappa_i} \le C_4 h, \qquad (27)$$

where

$$C_4 = \frac{C_3}{\min_j (\alpha_j - \tau)}.$$

Then from (25) - (27) we have

$$\max_{\Omega_h} | \varepsilon_{ij}^2 | \le C_5 h, \quad C_5 = \max_i \tau_i C_4. \qquad (28)$$

Based on (20), (24) and (28) we have

$$\max_{\bar{\Omega}_h} | \varepsilon_{ij} | \le C_6 h, \qquad (29)$$

where $C_6 = C_1 + C_5$.

Theorem 1 is proved.

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