Original Article

INVESTIGATING THE SOLUTION OF A BOUNDARY PROBLEM

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Abstract

For the elliptic type equation of the first form, the solution of the boundary problem in different regions was investigated. Here, the solution of a boundary value problem in the half-plane for the Cauchy-Riemann equation is investigated. The boundary condition is given only on the boundary that is identical to the second half-plane. Note that this condition has a special form. So, the Karleman condition is met for two points moving at the same time on this boundary. The examination of the solution of the considered boundary issue is based on the properties of the necessary conditions obtained in the case. The singularities in the necessary conditions are regularized by means of the given boundary condition.

Keywords: Cauchy-Riemann equation, Steklov problem, necessary conditions, singularity, regularization, Fredholm, eigenvalues, eigenfunctions.

1. Introduction.

It is known that the Steklov problem is understood as a spectral problem in which the spectral parameter is included only in the boundary condition [2]- [4]. Problems of this type can be considered both for ordinary differential equations and for special differential equations. So, for the ordinary differential equation, the spectrum can be finite in general in such problems [9]. In Steklov problems for special differential equations, problems for elliptic type equations are considered [5]-[7]. We will consider here the Steklov problem for the Cauchy-Riemann equation, which is a differential equation of first order elliptic type. The matter was considered in the upper half-level. Note that the provision of a non-local boundary condition for the Cauchy-Riemann equation in a limited plane domain with two points (ie, if two points move on the boundary at the same time) is for the Fredholm of this problem, i.e. for bringing this boundary problem to a system of Fredholm equations of the second type without a singularity at the core has been shown to be sufficient. The planar domain considered here and its boundary are unlimited. Since the solution of the Cauchy-Riemann equation is an analytic function, the solution cannot be taken to be zero in the part of the boundary at infinity, because then the solution can be zero as an identity. On the other hand, if the Dirichle condition is given on the entire boundary, then the solution of this problem may not exist in the general case [8]. Therefore, we will give this problem without any condition on the boundary of the half-plane at infinity, but a special boundary condition on the main boundary (being a two-point problem, the Carleman condition is satisfied where , and are generally complex numbers, and is the spectral parameter. It is known that the fundamental solution of the Cauchy-Riemann equation

2. Setting the issue.

Let's look at the issue as follows:

$$\frac{\partial u(x)}{\partial x_2}+i\frac{\partial u(x)}{\partial x_1}=0, \quad x_1\in R, \quad x_2>0 \tag{2.1}$$

$$\alpha_1 u(-t,0)+\lambda\alpha_2 u(t,o)=0, \quad t\geq 0, \tag{2.2}$$

where, $i=\sqrt{-1}$, α_1 and α_2 are generally complex numbers, λ and is the spectral parameter. It is known that the fundamental solution of the Cauchy-Riemann equation[1]

$$U(x-\xi)=\frac{1}{2\pi}\cdot\frac{1}{x_2-\xi_2+i(x_1-\xi_1)}, \tag{2.3}$$

With the help of this fundamental solution, we construct the second Green's formula for equation (2.1) in the upper half-plane. In other words, let's multiply both sides of the equation (2.1) by the fundamental solution (2.3) and integrate over the upper half-plane:

$$0=\int_R dx_1\int_0^\infty \frac{\partial u}{\partial x_2}U(x-\xi)dx_2+i\int_0^\infty dx_2\int_R \frac{\partial u}{\partial x_1}U(x-\xi)dx_1=0.$$

Let's integrate the integrals in the resulting expression piece by piece:

$$0=\int_R dx_1\left[u(x)U(x-\xi)\Big/_{x_2=0}^{\infty}-\int_0^\infty u(x)\frac{\partial U(x-\xi)}{\partial x_2}dx_2\right]+i\int_0^\infty dx_2\left[u(x)U(x-\xi)\Big/_{x_1=-\infty}^{\infty}-\right.$$

$$\left.-\int_R u(x)\frac{\partial U(x-\xi)}{\partial x_1}dx_1\right]=\lim_{x_2\to\infty}\int_R u(x)U(x-\xi)dx_1-\int_R u(x_1,0)U(x_1-\xi_1-\xi_2)dx_1-$$

$$-\int_R dx_1\int_0^\infty u(x)\frac{\partial U(x-\xi)}{\partial x_2}dx_2+i\lim_{x_1\to\infty}\int_0^\infty u(x)U(x-\xi)dx_2-i\lim_{x_1\to-\infty}\int_0^\infty u(x)U(x-\xi)dx_2-$$

$$-i\int_0^\infty dx_2\int_R u(x)\frac{\partial U(x-\xi)}{\partial x_1}dx_1,$$

Integrating the integrals along the half-plane, considering that (2.3) is the fundamental solution for (2.1), i.e.

$$\frac{\partial u(x-\xi)}{\partial x_2}+i\frac{\partial u(x-\xi)}{\partial x_1}=\delta(x-\xi)=\delta(x_1-\xi_1)\delta(x_2-\xi_2), \tag{2.4}$$

where the two-dimensional delta function is Dirac. Thus we get the basic relation as follows.

$$\lim_{x_2\to\infty}\int_R u(x)U(x-\xi)dx_1-\int_R u(x_1,0)U(x_1-\xi_1-\xi_2)dx_1+\lim_{x_1\to\infty}\int_0^\infty u(x)U(x-\xi)dx_2-$$
$$-i\lim_{x_1\to-\infty}\int_0^\infty u(x)U(x-\xi)dx_2=\int_R dx_1\int_0^\infty u(x)\left[\frac{\partial U(x-\xi)}{\partial x_2}+i\frac{\partial U(x-\xi)}{\partial x_1}\right]dx_2=$$
$$=\int_R dx_1\int_0^\infty u(x)\delta(x-\xi)dx_2=\begin{cases}u(\xi),\ \xi_1\in R,\ \xi_2>0,\\ \frac{1}{2}u(\xi),\ \xi_1\in R,\ \xi_2=0;\ \xi_1\in R,\ \xi_2=\infty;\ \xi_2\geq 0,\ \xi_1=-\infty;\ \xi_2\geq 0,\xi_1=\infty.\end{cases}$$
2.5)

The first part of the main relation we get gives the arbitrary solution determined in the domain of (2.1), and the second expression gives the necessary conditions.

3. Prerequisites.

Let us distinguish the necessary conditions included in (2.5), which is the main relation.

$$\frac{1}{2}u(\xi_1,0)=\frac{1}{2\pi}\int_R\frac{u(x_1,\infty)}{\infty+i(x_1-\xi_1)}dx_1-\frac{1}{2\pi}\int_R\frac{u(x_1,0)}{i(x_1-\xi_1)}dx_1+$$
$$+\frac{i}{2\pi}\int_0^\infty\frac{u(\infty,x_2)}{x_2+i(\infty-\xi_1)}dx_2-\frac{i}{2\pi}\int_0^\infty\frac{u(-\infty,x_2)}{x_2+i(-\infty-\xi_1)}dx_2,\quad \xi_1\in R, \tag{3.1}$$

$$\frac{1}{2}u(\xi_1,\infty)=\frac{1}{2\pi}\int_R\frac{u(x_1,\infty)}{i(x_1-\xi_1)}dx_1-\frac{1}{2\pi}\int_R\frac{u(x_1,0)}{-\infty+i(x_1-\xi_1)}dx_1+$$
$$+\frac{i}{2\pi}\int_0^\infty\frac{u(\infty,x_2)}{x_2-\infty+i(\infty-\xi_1)}dx_2-\frac{i}{2\pi}\int_0^\infty\frac{u(-\infty,x_2)}{x_2-\infty+i(-\infty-\xi_1)}dx_2,\quad \xi_1\in R, \tag{3.2}$$

$$\frac{1}{2}u(-\infty,\xi_2)=\frac{1}{2\pi}\int_R\frac{u(x_1,\infty)}{\infty-\xi_2+i(x_1+\infty)}dx_1-\frac{1}{2\pi}\int_R\frac{u(x_1,0)}{-\xi_2+i(x_1+\infty)}dx_1+$$
$$+\frac{i}{2\pi}\int_0^\infty\frac{u(\infty,x_2)}{x_2-\xi_2+i(\infty+\infty)}dx_2-\frac{i}{2\pi}\int_0^\infty\frac{u(-\infty,x_2)}{x_2-\xi_2)}dx_2,\quad \xi_2\geq 0, \tag{3.3}$$

$$\frac{1}{2}u(\infty,\xi_2)=\frac{1}{2\pi}\int_R\frac{u(x_1,\infty)}{\infty-\xi_2+i(x_1-\infty)}dx_1-\frac{1}{2\pi}\int_R\frac{u(x_1,0)}{-\xi_2+i(x_1-\infty)}dx_1+$$

$$+\frac{i}{2\pi}\int_0^\infty\frac{u(\infty,x_2)}{x_2-\xi_2}dx_2-\frac{i}{2\pi}\int_0^\infty\frac{u(-\infty,x_2)}{x_2-\xi_2+(-\infty-\infty)}dx_2,\quad \xi_2\geq 0, \tag{3.4}$$

Thus, we get the following verdict.

Theorem 1. Boundary values of an arbitrary function analytic in the upper half-plane satisfy relations (3.1)-(3.4).

Note 1. If the function, which is analytic in the upper half-plane, is bounded in the entire closed half-plane, then the necessary conditions (3.1)-(3.4) we gave above fall into the following form.

$$u(\xi_1,0)=\frac{i}{\pi}\int_R u(x_1,0)\frac{dx_1}{x_1-\xi_1}, \tag{3.11}$$

$$u(\xi_1,\infty)=-\frac{i}{\pi}\int_R u(x_1,\infty)\frac{dx_1}{x_1-\xi_1}, \tag{3.21}$$

$$u(-\infty,\xi_2)=-\frac{i}{\pi}\int_R u(-\infty,x_2)\frac{dx_2}{x_2-\xi_2}, \tag{3.31}$$

$$u(\infty,\xi_2)=\frac{i}{\pi}\int_0^\infty u(\infty,x_2)\frac{dx_2}{x_2-\xi_2} \tag{3.41}$$

Returning again to the basic relation (2.5), we get from there for the bounded solution in the closed upper half-plane:

$$u(\xi)=-\frac{1}{2\pi}\int_R\frac{u(x_1,0)}{-\xi_2+i(x_1-\xi_1)}dx_1=\frac{1}{2\pi}\int_R\frac{u(x_1,0)}{\xi_2-i(x_1-\xi_1)}dx_1,\ \ \xi_1\in R,\ \xi_2>0. \tag{3.5}$$

Thus, it can be seen from (3.5) that in order to find a bounded solution in the upper half-plane (closed), it is enough to know , but it can be seen from (3.11) that , cannot be given arbitrarily. So, the necessary condition (3.11) must be satisfied.

Examining the necessary condition. Let us now divide the necessary condition (3.11) into two parts as follows.

$$u(-\xi_1,0)=\frac{i}{\pi}\int_{-\infty}^{0}\frac{u(x_1,0)}{x_1+\xi_1}dx_1+\frac{i}{\pi}\int_{0}^{\infty}\frac{u(x_1,0)}{x_1+\xi_1)}dx_1=\frac{-i}{\pi}\int_{\infty}^{0}\frac{u(-x_1,0)}{-x_1+\xi_1}dx_1+$$

$$+\frac{i}{\pi}\int_{0}^{\infty}\frac{u(x_1,0)}{x_1+\xi_1}dx_1=\frac{-i}{\pi}\int_{0}^{\infty}\frac{u(-x_1,0)}{x_1-\xi_1}dx_1+\frac{i}{\pi}\int_{0}^{\infty}\frac{u(x_1,0)}{x_1+\xi_1}dx_1 \quad \xi_1\geq 0, \tag{3.6}$$

$$u(\xi_1,0)=\frac{i}{\pi}\int_{-\infty}^{0}\frac{u(x_1,0)}{x_1-\xi_1}dx_1+\frac{i}{\pi}\int_{0}^{\infty}\frac{u(x_1,0)}{x_1-\xi_1}dx_1=\frac{-i}{\pi}\int_{\infty}^{0}\frac{u(-x_1,0)}{-x_1-\xi_1}dx_1+$$

$$+\frac{i}{\pi}\int_{0}^{\infty}\frac{u(x_1,0)}{x_1-\xi_1}dx_1=\frac{-i}{\pi}\int_{0}^{\infty}\frac{u(-x_1,0)}{x_1+\xi_1}dx_1+\frac{i}{\pi}\int_{0}^{\infty}\frac{u(x_1,0)}{x_1-\xi_1}dx_1 \quad \xi_1\geq 0. \tag{3.7}$$

Here, taking into account the boundary condition (2.2), we construct the following linear combination:

$$\alpha_1 u(-\xi_1,0)-\lambda\alpha_2 u(\xi_1,0)=\frac{i}{\pi}\int_{0}^{\infty}\frac{\alpha_1 u(x_1,0)+\lambda\alpha_2 u(-x_1,0)}{x_1+\xi_1}dx_1-$$

$$\frac{i}{\pi}\int_{0}^{\infty}\frac{\alpha_1 u(-x_1,0)+\lambda\alpha_2 u(x_1,0)}{x_1-\xi_1}dx_1=\frac{i}{\pi}\int_{0}^{\infty}\frac{\alpha_1 u(x_1,0)+\lambda\alpha_2 u(-x_1,0)}{x_1+\xi_1}dx_1 \tag{3.8}$$

So we get.

Theorem 2. The regular relation (3.8) is satisfied for the bounded solution of the given boundary problem (2.1)-(2.2).

4. Fredholm.

If we connect the obtained regular expression (3.8) to the given boundary condition (2.2):

$$\begin{cases}\alpha_1 u(-t,0)+\lambda\alpha_2 u(t,0)=0,\\ \alpha_1 u(-t,0)-\lambda\alpha_2 u(t,0)=\dfrac{i}{\pi}\displaystyle\int_{0}^{\infty}\frac{\alpha_1 u(\tau,0)+\lambda\alpha_2 u(-\tau,0)}{\tau+t}d\tau.\end{cases} \tag{4.1}$$

If so, then from (4.1) we get:

$$\begin{cases} u(-t,0) = \dfrac{i}{2\pi}\displaystyle\int_0^\infty \dfrac{u(\tau,0)}{\tau+t}d\tau + \dfrac{\lambda\alpha_2 i}{2\pi\alpha_1}\displaystyle\int_0^\infty \dfrac{u(-\tau,0)}{\tau+t}d\tau, \\ u(t,0) = -\dfrac{\alpha_1 i}{2\lambda\alpha_2\pi}\displaystyle\int_0^\infty \dfrac{u(\tau,0)}{\tau+t}d\tau - \dfrac{i}{2\pi}\displaystyle\int_0^\infty \dfrac{u(-\tau,0)}{\tau+t}d\tau, \ \ t \geq 0. \end{cases} \tag{4.2}$$

With this, we get the following verdict.

Theorem 3. If (2.1)-(2.2) then the bounded solution of the boundary problem in the closed upper half-plane is of the form (3.5), and the value of (4.2) is determined from the system of regular integral equations of the second type. Finally, if we use the boundary condition (2.2) again, we get the system (4.2).

$$u(-t,0) = \frac{i}{2\pi}\int_0^\infty \frac{d\tau}{\tau+t}\cdot\frac{-\alpha_1 u(-\tau,0)}{\lambda\alpha_2} + \frac{\lambda\alpha_2 i}{2\pi\alpha_1}\int_0^\infty \frac{u(-\tau,0)}{\tau+t}d\tau =$$
$$= \frac{1}{2\pi i}\int_0^\infty \left(\frac{\alpha_1}{\lambda\alpha_2} - \frac{\lambda\alpha_2}{\alpha_1}\right)\frac{u(-\tau,0)}{\tau+t}d\tau, \tag{4.3}$$

$$u(t,0) = \frac{\alpha_1}{\lambda\alpha_2 2\pi i}\int_0^\infty \frac{u(\tau,0)}{\tau+t}d\tau + \frac{1}{2\pi i}\int_0^\infty \frac{d\tau}{\tau+t}\cdot\frac{-\lambda\alpha_2 u(\tau,0)}{\alpha_1} =$$
$$= \frac{1}{2\pi i}\int_0^\infty \left(\frac{\alpha_1}{\lambda\alpha_2} - \frac{\lambda\alpha_2}{\alpha_1}\right)\frac{u(\tau,0)}{\tau+t}d\tau, \tag{4.4}$$

Since the kernels of the integral equations (4.3) and (4.4) we obtained are the same, it is enough to look at the following equation.

$$y(t) = \rho\int_0^\infty \frac{y(\tau)}{\tau+t}d\tau, \quad t > 0, \tag{4.5}$$

Where,

$$\rho = \frac{1}{2\pi i}\left(\frac{\alpha_1}{\lambda\alpha_2} - \frac{\lambda\alpha_2}{\alpha_1}\right), \tag{4.6}$$

and the new parameter y(t) is in (4.3) and (4.4).

5. Approximate calculation of special numbers and functions.

Let us discretize the equation (4.5) obtained above [10].

$$y(t)=\rho\sum_{k=0}^{\infty}\int_{k}^{k+1}\frac{y(\tau)}{\tau+t}d\tau,$$

let's apply the method of rectangles to the integrals on the right side of the expression.

$$y(t)=\rho\sum_{k=0}^{\infty}\frac{y(k+\frac{1}{2})}{k+\frac{1}{2}+t},$$

Or

$$y(n+\frac{1}{2})=\rho\sum_{k=0}^{\infty}\frac{y(k+\frac{1}{2})}{k+n+1},\quad n\geq 0. \tag{5.1}$$

Finally

$$y(n+\frac{1}{2})=z_n \quad n\geq 0, \tag{5.2}$$

if we adopt the notation:

$$z_n=\rho\sum_{k=0}^{\infty}\frac{z_k}{k+n+1},\quad n\geq 0. \tag{5.3}$$

Let's write this system explicitly:

$$(\rho-1)z_0+\rho\frac{z_1}{2}+\rho\frac{z_2}{3}+\rho\frac{z_3}{4}+\ldots=0,$$

$$\rho\frac{z_0}{2}+(\rho\frac{1}{3}-1)z_1+\rho\frac{z_2}{4}+\rho\frac{z_3}{5}+\ldots=0,$$

$$\rho\frac{z_0}{3}+\rho\frac{z_1}{4}+(\rho\frac{1}{5}-1)z_2+\rho\frac{z_3}{6}+\ldots=0,$$

....

$$\rho\frac{z_0}{m}+\rho\frac{z_1}{m+1}+\ldots+\rho\frac{z_{m-2}}{2m-2}+(\rho\frac{1}{2m-1}-1)z_{m-1}+\rho\frac{z_m}{2m}+\ldots=0$$

.... (5.4)

Consider the following determinant:

$$\Delta_n = \begin{vmatrix} (\rho-1) & \frac{\rho}{2} & \frac{\rho}{3} \cdots\cdots\cdots & \frac{\rho}{n} \\ \frac{\rho}{2} & \frac{\rho}{3}-1 & \frac{\rho}{4} \cdots\cdots\cdots & \frac{\rho}{n+1} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \frac{\rho}{n} & \frac{\rho}{n+1} & \frac{\rho}{n+2} \cdots\cdots\cdots & \frac{\rho}{2n-1}-1 \end{vmatrix} = 0. \qquad (5.5)$$

Let us denote the roots of this equation. They are approximate values for the eigenvalues of (4.5). This value is written in (4.6), and the -s determined from there are approximate values for specific numbers of the given problem. The approximate expression for the specific functions of the considered boundary problem (2.1)-(2.2) is obtained from (3.5). For this, we write in (4.5) and get an approximate expression for it. Instead of this expression, (3.5) is also written. Thus, a scheme for the approximate calculation of eigenvalues and functions was shown.

Note 2. If the eigenvalues and functions from (4.5) are found exactly, then the eigenvalues and functions of (2.1)-(2.2) can also be determined exactly.

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