Original Article

Sum Degree Divided By Diameter Energy Of Graph

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Abstract

In this paper, we introduce the concept of sum degree divided by diameter matrix $\frac{SD}{diam}(\Gamma)$ of the graph Γ and obtain the characteristic polynomial and energy. We also find some bounds, spectra and energy of different class of graphs. In [5],[6] the concept of product degree divided by diameter energy of graphs and is introduced.

Keywords: Sum degree, Diameter, Spectrum and Energy

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1 Introduction

Let Γ be a finite and undirected simple graph on n vertices named by $v_1, v_2, \ldots, v_n$. The adjacency matrix $A(\Gamma)$ of the graph Γ is a square matrix of order n, whose (i,j) -entry is equal to 1 if the vertices v_i and v_j are adjacent and is equal to 0 otherwise. The characteristic polynomial of the adjacency matrix, i.e., $\det(xI_P - A(\Gamma))$, where I is the unit matrix of order n, is said to be the characteristic polynomial of the graph Γ and will be denoted by $\phi(\Gamma, x)$ .The eigenvalue of a graph Γ is defined as the eigenvalues of its adjacency matrix $A(\Gamma)$, and so they are just the roots of the equation $\phi(\Gamma, x) = 0$ since $A(\Gamma)$ is a real symmetric matrix, so its eigenvalues are all real. Denoting them by $\psi_1, \psi_2, \ldots, \psi_p$ and as a whole, they are called the spectrum of Γ. Spectral properties of graphs, including properties of the characteristic polynomial, have been extensively studied, for details, we refer to [2]. In the 1970s, I. Gutman in [3] introduced the concept of the energy of Γ .

2 Sum degree divided by diameter matrix and its energy

Let $\Gamma = (V, E)$ be a connected simple graph with $|V| = p$ vertices and $|E| = q$ edges and let d_i denote the degree of the vertex v_i, for $i = 1,2, \ldots\ldots, p$. For vertices $v_i, v_j \in V(\Gamma)$, the distance $d(v_i, v_j)$ is defined as the length of a shortest path between v_i and v_j in Γ [1]. The eccentricity of a vertex is the maximum distance from it to any other vertex, $\varepsilon_i = \max_{v_j \in V(\Gamma)} d(v_i, v_j)$.

The diameter of a graph $diam(\Gamma)$ is the maximum eccentricity of any vertex in the graph, or the greatest distance between any pair of vertices.

Let d_i, d_j be the degrees of the vertices v_i, v_j respectively for all $i, j = 1,2,3, \ldots, p$ then the sum degree by diameter matrix of the graph is defined as,

$$b_{ij} = \begin{cases} \frac{d_i + d_j}{diam(\Gamma)} & \text{if there is a path between } v_i \text{ and } v_j \\ 0 & \text{otherwise} \end{cases}.$$

The sum degree divided by diameter matrix is a symmetric matrix with eigen values as $\psi_1 \geq \psi_2 \geq \psi_3 \geq \ldots\ldots\ldots\ldots \geq \psi_p$.

The characteristic polynomial of $\frac{SD}{diam}(\Gamma)$ is given by $\det\left|\psi I - \frac{SD}{diam}(\Gamma)\right|$.
The Sum degree divided by diameter energy of the graph Γ is defined as sum of absolute values of $\psi_i, i = 1,2,\ldots\ldots, p$.

$$E\left[\frac{SD}{diam}(\Gamma)\right] = \sum_{i=1}^{p} |\psi_i|.$$

3 Properties of Sum Degree divided by diameter energy

Theorem 3.1 If eigen values of $\frac{SD}{diam}(\Gamma)$ are $\psi_1^+ > \psi_2^+ > \cdots > \psi_r^+$, then

1. $\sum_{i=1}^{p} \psi_i = 0$ and
2. $\sum_{i=1}^{p} \psi_i^2 = 2\sum_{i=1}^{p} \left(\frac{d_i+d_j}{diam(\Gamma)}\right)^2 = 2\Phi$

where $\Phi = \sum_{i=1}^{p} \left(\frac{d_i+d_j}{diam(\Gamma)}\right)^2$.

Proof. (1) Since the diagonal entries are zero the sum of leading diagonal entries of $\frac{SD}{diam}(\Gamma)$ is zero .
Hence $\sum_{i=1}^{p} \psi_i = 0$.
(2) The sum of squares of latent roots of $\frac{SD}{diam}(\Gamma)$ is the sum of latent roots of $\left[\frac{SD}{diam}(\Gamma)\right]^2$,
$\sum_{i=1}^{n} \psi_i^2 = \sum_{i=1}^{p} \sum_{j=1}^{n} u_{ij}\, u_{ji}$
$= 0 + 2\sum_{i<j} (u_{ij})^2$
$= 2\sum_{i=1}^{p} \left(\frac{d_i+d_j}{diam(\Gamma)}\right)^2$
$= 2\Phi$.

Theorem 3.2 If c_0, c_1 and c_2 are the first three coefficients of characteristic polynomial of $\frac{SD}{diam}(\Gamma)$ matrix, then

1. $c_0 = 1$,
2. $c_1 = 0$ and
3. $c_2 = -\Phi$.

Proof. (i)By definition, $\Gamma(\psi, x) = \det[\psi I - \Phi]$.
Therefore $c_0 = 1$.
(ii) $c_1 = (-1)^1 \times \mathrm{trace}(\Gamma) = -1 \times 0 = 0$.
(iii) By definition $c_2 = \sum_{1\le i<j\le p} \begin{vmatrix} u_{ii} & u_{ij} \\ u_{ji} & u_{jj} \end{vmatrix} = \sum_{1\le i<j\le p} (a_{ii}u_{jj} - u_{ij}u_{ji})$
$= \sum_{1\le i<j\le p} u_{ii}u_{jj} - \sum_{1\le i<j\le p} a_{ij}^2 = 0 - \Phi = -\Phi$.

We have the following bounds for $\frac{SD}{diam}(\Gamma)$ using McClelland's inequalities .

Theorem 3.3 Let Γ be a graph with p vertices, then the upper bound for $\frac{SD}{diam}(\Gamma)$ is

$$E\left[\frac{SD}{diam}(\Gamma)\right] \le \sqrt{2p\Phi}.$$

Proof. Let $\psi_1 \geq \psi_2 \geq \cdots \geq \psi_p$ be the eigen values of $\frac{SD}{diam}(\Gamma)$, then by Using Cauchy-Schwarz inequality we have,

$$\left[\sum_{i=1}^{p} u_i v_i\right]^2 \leq \left[\sum_{i=1}^{p} u_i^2\right]\left[\sum_{i=1}^{p} v_i^2\right].$$

Choose $u_i = 1, v_i = |\psi_i|$ and by Theorem 3.1

$$\left[\sum_{i=1}^{p} |\psi_i^+|\right]^2 \leq \left[\sum_{i=1}^{p} 1\right]\left[\sum_{i=1}^{p} |\psi_i^+|^2\right] = p\sum_{i=1}^{p} {\psi_i^+}^2$$

$$\left[\left(\frac{SD}{diam}\right) E(\Gamma)\right]^2 \leq p2\Phi.$$

Hence

$$E\left[\frac{SD}{diam}(\Gamma)\right] \leq \sqrt{2p\Phi}.$$

We present the following lower bounds for $E\left[\frac{SD}{diam}(\Gamma)\right]$.

Theorem 3.4 Let G be a graph with p vertices. If $\tau = \left|\det\frac{SD}{diam}(\Gamma)\right|$ of Γ, then the lower bound is

$$E\left[\frac{SD}{diam}(\Gamma)\right] \geq \sqrt{2\Phi + p(p-1)\tau^{\frac{2}{p}}}.$$

Proof. By definition we have,

$$\left[E\left(\frac{SD}{diam}(\Gamma)\right)\right]^2 = \left[\sum_{i=1}^{p} |\psi_i|\right]^2 = \left[\sum_{i=1}^{p} |\psi_i|\right]\left[\sum_{j=1}^{p} |\psi_j|\right]$$

$$= \sum_{i=1}^{p} |\psi_i|^2 + \sum_{i\neq j} |\psi_i||\psi_j|.$$

From the inequality of arthimetic and geometric means

$$\frac{1}{p(p-1)}\sum_{i\neq j} |\psi_i||\psi_j| \geq \left[\prod_{i\neq j} |\psi_i||\psi_j|\right]^{\frac{1}{p(p-1)}}.$$

Therefore

$$\left[E\left(\frac{SD}{diam}(\Gamma)\right)\right]^2 \geq \sum_{i=1}^{p} |\psi_i|^2 + p(p-1)\left[\prod_{i\neq j} |\psi_i||\psi_j|\right]^{\frac{1}{p(p-1)}}$$

$$\geq \sum_{i=1}^{p} |\psi_i|^2 + p(p-1)\left[\prod_{i=1}^{p} |\psi_i|^{2(p-1)}\right]^{\frac{1}{p(p-1)}}.$$

$$= \sum_{i=1}^{p} |\psi_i|^2 + p(p-1)\left|\prod_{i=1}^{p} \psi_i\right|^{\frac{2}{p}}$$

$$= 2\Phi + p(p-1)\tau^{\frac{2}{p}}.$$

Hence

$$E\left[\frac{SD}{diam}(\Gamma))\right] \geq \sqrt{2\Phi + p(p-1)\tau^{\frac{2}{p}}}.$$

Theorem 3.5 Let r_i and s_i, $1 \le i \le p$ be positive real numbers with $M_1 = \max_{1\le i\le p}(r_i)$, $M_2 = \max_{1\le i\le p}(s_i)$, $m_1 = \min_{1\le i\le p}(r_i)$, $m_2 = \min_{1\le i\le n}(s_i)$ then by theorem 90 of [4]

$$\sum_{i=1}^{p} r_i^2 \sum_{i=1}^{p} s_i^2 \le \frac{1}{4}\left[\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right]^2 \left[\sum_{i=1}^{n} r_is_i\right]^2.$$

Theorem 3.6 For a graph Γ with p vertices let $|\psi_1|$ and $|\psi_p|$ are the maximum and minimum eigen values among all $|\psi_i|'s$ of $\frac{SD}{diam}(\Gamma)$ respectively, then we have

$$E\left[\frac{SD}{diam}(\Gamma)\right] \ge \frac{\sqrt{8p\Phi|\psi_1||\psi_n|}}{|\psi_1|+|\psi_p|}.$$

Proof. Consider a graph Γ with p vertices let $|\psi_1|$ and $|\psi_p|$ are the maximum and minimum eigen values among all $|\psi_i|'s$ of $\frac{SD}{diam}(\Gamma)$ respectively.

From theorem 3.5,

$$\sum_{i=1}^{p} r_i^2 \sum_{i=1}^{p} s_i^2 \le \frac{1}{4}\left[\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right]^2 \left[\sum_{i=1}^{p} r_is_i\right]^2.$$

Let $r_i = 1, s_i = |\zeta_i|, M_1M_2 = |\psi_i|, m_1m_2 = |\psi_p|$ then

$$\sum_{i=1}^{p} 1^2 \sum_{i=1}^{p} \psi_i^2 \le \frac{1}{4}\left[\sqrt{\frac{|\psi_1|}{|\psi_p|}} + \sqrt{\frac{|\psi_n|}{|\psi_1|}}\right]^2 \left[\sum_{i=1}^{p} 1|\psi_i|\right]^2$$

From theorem 3.1

$$p2\Phi \le \frac{1}{4}\left[\frac{(|\psi_1|+|\psi_p|)^2}{|\psi_1||\psi_n|}\right]\left[E\left(\frac{SD}{diam}(\Gamma)\right)\right]^2,$$

$$\left[E\left(\frac{SD}{diam}(\Gamma)\right)\right]^2 \ge \frac{8p\Phi|\psi_1||\psi_p|}{(|\psi_1|+|\psi_p|)^2}$$

$$E\left[\frac{SD}{diam}(\Gamma)\right] \ge \frac{\sqrt{8p\Phi|\psi_1||\psi_p|}}{|\psi_1|+|\psi_p|}.$$

Theorem 3.7 Let r_i and s_i, $1 \le i \le n$ be non negative real numbers with $M_1 = \max_{1\le i\le n}(r_i)$, $M_2 = \max_{1\le i\le n}(s_i)$, $m_1 = \min_{1\le i\le n}(r_i)$, $m_2 = \min_{1\le i\le n}(s_i)$ then by theorem 3.1 of [8]

$$\sum_{i=1}^{n} r_i^2 \sum_{i=1}^{n} s_i^2 - \left[\sum_{i=1}^{n} r_is_i\right]^2 \le \frac{n^2}{4}(M_1M_2 - m_1m_2)^2.$$

Theorem 3.8 For a graph G with p vertices, we have

$$E\left[\frac{SD}{diam}(\Gamma)\right] \ge \sqrt{2p\Phi - \frac{p^2}{4}\left(|\psi_1| - |\psi_p|\right)^2}.$$

Proof. Consider a graph Γ with p vertices let $|\psi_1|$ and $|\psi_p|$ are the maximum and minimum eigen values among all $|\psi_i|'s$ of $\frac{SD}{diam}(\Gamma)$ respectively.

From theorem 3.7,

$$\sum_{i=1}^{p} r_i^2 \sum_{i=1}^{p} s_i^2 - \left[\sum_{i=1}^{p} r_i s_i\right]^2 \leq \frac{p^2}{4}(M_1M_2 - m_1m_2)^2.$$

Let $r_i = 1, s_i = |\psi_i|, M_1M_2 = |\psi_i|, m_1m_2 = |\psi_p|$, then

$$\sum_{i=1}^{p} 1^2 \sum_{i=1}^{p} \psi_i^2 - \left[\sum_{i=1}^{p} 1|\psi_i|\right]^2 \leq \frac{p^2}{4}\left(|\psi_1| - |\zeta_p|\right)^2.$$

From theorem 3.1

$$p2\Phi - \left[E\left(\frac{SD}{diam}(\Gamma)\right)\right]^2 \leq \frac{p^2}{4}(|\psi_1| - |\psi_n|)^2,$$

$$E\left[\frac{SD}{diam}(\Gamma)\right] \geq \sqrt{2p\Phi - \frac{p^2}{4}\left(|\psi_1| - |\psi_p|\right)^2}.$$

Theorem 3.9 Let r_i and s_i, $1 \leq i \leq p$ be positive real numbers, then by [3]

$$|p\sum_{i=1}^{p} r_i s_i - \sum_{i=1}^{p} r_i \sum_{i=1}^{p} s_i| \leq \mu(p)(A - a)(B - b)$$

where a, b, A and B are real constants, that for each $i, 1 \leq i \leq p, a \leq a_i \leq A$ and $b \leq b_i \leq B$. Further, $\mu(p) = p\lfloor\frac{p}{2}\rfloor\left(1 - \frac{1}{p}\lfloor\frac{p}{2}\rfloor\right)$.

Theorem 3.10 For a graph Γ with p vertices, we have

$$E\left[\frac{SD}{diam}(\Gamma)\right] \geq \sqrt{2p\Phi - \mu(p)\left(|\psi_1| - |\psi_p|\right)^2}.$$

Proof. Consider a graph Γ with p vertices let $|\psi_1|$ and $|\psi_p|$ are the maximum and minimum eigen values among all $|\psi_i|'s$ of $\frac{SD}{diam}(\Gamma)$ respectively.

From theorem 3.9,

$$|n\sum_{i=1}^{p} r_i s_i - \sum_{i=1}^{p} r_i \sum_{i=1}^{p} s_i| \leq \mu(p)(A - a)(B - b).$$

Let $r_i = s_i = |\psi_i|, A = B = |\psi_1|, a = b = |\psi_n|$ then

$$|p\sum_{i=1}^{p} |\psi_i|^2 - \left[\sum_{i=1}^{p} |\psi_i|\right]^2| \leq \mu(p)\left(|\psi_1| - |\psi_p|\right)\left(|\psi_1| - |\psi_p|\right).$$

From theorem 3.1

$$|p2\Phi - \left[E\left(\frac{SD}{diam}(\Gamma)\right)\right]^2| \leq \mu(p)\left(|\psi_1| - |\psi_p|\right)^2,$$

$$E\left[\frac{SD}{diam}(\Gamma)\right] \geq \sqrt{2p\Phi - \mu(p)\left(|\psi_1| - |\psi_p|\right)^2}.$$

4 Sum degree divided by diameter matrix and its energy for standard graphs

Theorem 4.1 Let K_p be a complete graph with p vertices, then

$$E\left[\frac{SD}{diam}(K_p)\right] = 4(p^2 - 2p + 1).$$

Proof. The complete graph K_p with n-vertices have their sum degree by diameter matrix as follows

$$\frac{SD}{diam}(K_p) = \begin{bmatrix} 0 & 2(p-1) & 2(p-1) & & 2(p-1) \\ 2(p-1) & 0 & 2(p-1) & & 2(p-1) \\ 2(p-1) & 2(p-1) & 0 & & 2(p-1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2(p-1) & 2(p-1) & & 2(p-1) & 0 \end{bmatrix}.$$

Its characteristic polynomial is,

$$[\psi - (2p^2 - 4p + 2)][\psi - 2(p-1)]^{(p-1)} = 0.$$

$$\text{Spectra}\left[\frac{SD}{diam}(K_p)\right] = \begin{pmatrix} (2p^2 - 4p + 2) & 2(p-1) \\ 1 & p-1 \end{pmatrix}.$$

Therefore

$$E\left[\frac{SD}{diam}(K_p)\right] = |(2p^2 - 4p + 2)|1 + |2(p-1)|(p-1)$$

$$= 4(p^2 - 2p + 1).$$

Theorem 4.2 Let K_p be a star graph with p vertices, then

$$E\left[\frac{SD}{diam}(K_{1,p-1})\right] = (p-2) + \sqrt{4p^4 - 64p^3 + 432.6808p^2 - 1337.4824p + 1551.0368}.$$

Proof. The star graph $K_{1,p-1}$ with p-vertices has it's sum degree by diameter matrix as follows,

$$\frac{SD}{diam}(K_{1,p-1}) = \begin{bmatrix} 0 & \frac{p}{2} & \frac{p}{2} & & \frac{p}{2} \\ \frac{p}{2} & 0 & 1 & & 1 \\ \frac{p}{2} & 1 & 0 & & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{p}{2} & 1 & & 1 & 0 \end{bmatrix}.$$

Its characteristic polynomial is

$$\left[\psi - (-2p^2 + 16p - 30) + \sqrt{4p^4 - 64p^3 + 432.6808p^2 - 1337.4824p + 1551.0368}\right]$$
$$\left[\psi - (-2p^2 + 16p - 30) - \sqrt{4p^4 - 64p^3 + 432.6808p^2 - 1337.4824p + 1551.0368}\right] [\psi - (-1)]^{p-2} = 0$$

$$\text{Spectra}\left[\frac{SD}{diam}(K_{1,p-1})\right] =$$

$$\begin{pmatrix} (-2p^2 + 16p - 30) + \sqrt{4p^4 - 64p^3 + 432.6808p^2 - 1337.4824p + 1551.0368} & 1 \\ (-2p^2 + 16p - 30) + \sqrt{4p^4 - 64p^3 + 432.6808p^2 - 1337.4824p + 1551.0368} & 1 \\ -1 & p-2 \end{pmatrix}.$$ Therefore

$$E\left[\frac{SD}{diam}(K_{1,p-1})\right] =$$

$$|(-2p^2 + 16p - 30) + \sqrt{4p^4 - 64p^3 + 432.6808p^2 - 1337.4824p + 1551.0368}|1 + |(-2p^2 + 16p - 30) - \sqrt{4p^4 - 64p^3 + 432.6808p^2 - 1337.4824p + 1551.0368}|1 + |1|(p-2)$$

$$= (p-2) + \sqrt{4p^4 - 64p^3 + 432.6808p^2 - 1337.4824p + 1551.0368}.$$

Theorem 4.3 Let $K_{p,p}$ be a double star graph with p vertices, then

$$E\left[\frac{SD}{diam}(K_{p,p})\right] = \frac{0.6982p^2 + 19.184p - 17.2084}{3}.$$

Proof. The double star graph $K_{p,p}$ with p-vertices has it's sum degree by diameter matrix as follows

$$\frac{SD}{diam}(K_{p,p}) = \begin{bmatrix} 0 & \frac{p+1}{3} & \frac{p+1}{3} & \frac{p+1}{3} & \frac{2p}{3} & \frac{p+1}{3} & & \frac{p+1}{3} \\ \frac{p+1}{3} & 0 & \frac{2}{3} & \frac{2}{3} & \frac{p+1}{3} & \frac{2}{3} & & \frac{2}{3} \\ \frac{p+1}{3} & \frac{2}{3} & 0 & \frac{2}{3} & \frac{p+1}{3} & \frac{2}{3} & & \frac{2}{3} \\ \frac{2p}{3} & \frac{p+1}{3} & \frac{p+1}{3} & 0 & \frac{p+1}{3} & \frac{p+1}{3} & & \frac{p+1}{3} \\ \frac{p+1}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 & \frac{2}{3} & & \frac{2}{3} \\ \frac{p+1}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{p+1}{3} & 0 & & \frac{2}{3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{p+1}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{p+1}{3} & \frac{2}{3} & & 0 \end{bmatrix}.$$

Its characteristic polynomial is

$$[\psi - (-(0.3491p^2 + 6.5920p - 5.6042))][\psi - (0.3491p^2 + 6.5920p - 5.6042)]$$
$$[\psi - (\tfrac{-2p}{3})][\psi - \tfrac{-2}{3}]^{(2p-3)} = 0.$$

$$\text{Spectra}\left[\frac{SD}{diam}(K_{p,p})\right] =$$

$$\begin{pmatrix} -(0.3491p^2+6.5920p-5.6042) & (0.3491p^2+6.5920p-5.6042) & \left(\frac{-2p}{3}\right) & \frac{-2}{3} \\ 1 & 1 & 1 & (2p-3) \end{pmatrix}.$$

Therefore

$$E\left[\frac{SD}{diam}(K_{p,p})\right] =$$

$$|-(0.3491p^2+6.5920p-5.6042)|1+|(0.3491p^2+6.5920p-5.6042)|1+\left|\frac{-2p}{3}\right|1+\left|\frac{-2}{3}\right|(2p-3)$$

$$=\frac{0.6982p^2+19.184p-17.2084}{3}.$$

Theorem 4.4 Let $S_p^0, p \geq 3$ be a crown graph with $2p$ vertices, then

$$E\left[\frac{SD}{diam}(S_p^0)\right]=\frac{4}{3}(2p^2-3p+1).$$

Proof. The crown graph S_p^0 with p-vertices has it's sum degree by diameter matrix as follows

$$\frac{SD}{diam}(S_p^0)=\begin{bmatrix} 0 & \frac{2(p-1)}{3} & \frac{2(p-1)}{3} & \cdots & \frac{2(p-1)}{3} \\ \frac{2(p-1)}{3} & 0 & \frac{2(p-1)}{3} & \cdots & \frac{2(p-1)}{3} \\ \frac{2(p-1)}{3} & \frac{2(p-1)}{3} & 0 & \cdots & \frac{2(p-1)}{3} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{2(p-1)}{3} & \frac{2(p-1)}{3} & \frac{2(p-1)}{3} & \cdots & 0 \end{bmatrix}.$$

Its characteristic polynomial is

$$\left[\psi-\frac{4p^2-6p+2}{3}\right]\left[\psi-\frac{(-2p+2)^2}{3}\right]^{2p-1}=0$$

$$\text{Spectra}\left[\frac{SD}{diam}(S_n^0)\right]=\begin{pmatrix} \frac{4p^2-6p+2}{3} & \frac{(-2p+2)}{3} \\ 1 & 2p-1 \end{pmatrix}.$$

Therefore

$$E\left[\frac{SD}{diam}(S_p^0)\right]=\left|\frac{4p^2-6p+2}{3}\right|(1)+\left|\frac{(-2p+2)}{3}\right|(2p-1)$$

$$=\frac{4}{3}(2p^2-3p+1)$$

Theorem 4.5 Let $K_{p\times 2}$ be a cocktail party graph with $2p$ vertices, then

$$E\left[\frac{SD}{diam}(K_{p\times 2})\right] = 4(2p^2 - 3p + 1).$$

Proof. The cocktail party graph $K_{p\times 2}$ with $2p$-vertices has it's sum degree by diameter matrix as follows

$$\frac{SD}{diam}(K_{n\times 2}) = \begin{bmatrix} 0 & 2(p-1) & 2(p-1) & \dots & 2(p-1) \\ 2(p-1) & 0 & 2(p-1) & \dots & 2(p-1) \\ 2(p-1) & 2(p-1) & 0 & \dots & 2(p-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2(p-1) & 2(p-1) & 2(p-1) & & 0 \end{bmatrix}.$$

Its characteristic polynomial is

$[\psi - (4p^2 - 6p + 2)][\psi - 2(n-1)]^{2p-1} = 0$

$$\text{Spectra}\left[\frac{SD}{diam}(K_{p\times 2})\right] = \begin{pmatrix} 4p^2 - 6p + 2 & 2(p-1) \\ 1 & 2p-1 \end{pmatrix}.$$

Therefore

$$E\left[\frac{SD}{diam}(K_p)\right] = |4p^2 - 6p + 2|1 + |2(p-1)|(2n-1)$$
$$= 4(2p^2 - 3p + 1).$$

Theorem 4.6 Let F_p be a Friendship graph with p vertices, then

$$E\left[\frac{SD}{diam}(F_p)\right] = 2(2p-1) + \sqrt{104p^2 - 216p + 196}.$$

Proof. The Friendship graph F_p with $2p + 1$ vertices has it's sum degree by diameter matrix as follows

$$\frac{SD}{diam}(F_p) = \begin{bmatrix} 0 & p+1 & p+1 & & p+1 \\ p+1 & 0 & 2 & & 2 \\ p+1 & 2 & 0 & & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p+1 & 2 & & 2 & 0 \end{bmatrix}.$$

Its characteristic polynomial is

$$\left[\psi - \left((4p-2) + \sqrt{104p^2 - 216p + 196}\right)\right]$$
$$\left[\psi - \left((4p-2) - \sqrt{104p^2 - 216p + 196}\right)\right][\psi + (-2)]^{2p-1} = 0$$

$$\text{Spectra}\left[\frac{SD}{diam}(F_p)\right] =$$

$$\begin{pmatrix} (4p-2)+\sqrt{104p^2-216p+196} & (4p-2)-\sqrt{104p^2-216p+196} & -2 \\ 1 & 1 & 2p-1 \end{pmatrix}.$$

Therefore, $E\left[\frac{SD}{diam}(F_p)\right] =$

$$\left|(4p-2)+\sqrt{104p^2-216p+196}\right|1+\left|(4p-2)-\sqrt{104p^2-216p+196}\right|1+|-2|(2p-1)$$

$$= 2(2p-1)+\sqrt{104p^2-216p+196}.$$

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