Original Article

On Invariant Submanifolds of SQ-Sasakian Manifolds

P. Somashekhara[1], M.S. Siddesha[2*], C.S. Bagewadi[3], M.M. Praveena[4]

[1]Department of Mathematics, Government First Grade College, Kadur- 577548, Chikkamagalur, Karnataka, INDIA, E-mail: somumathrishi@gmail.com

[2]Department of Mathematics, Jain Deemed-to-be-University, Global Campus-562112, Karnataka, INDIA, E-mail: mssiddesha@gmail.com

[3]Department of Mathematics, Kuvempu University, Shankaraghatta - 577 451, Shimoga, Karnataka, INDIA, E-mail: prof_bagewadi@yahoo.co.in

[4]Department of Mathematics, M.S. Ramaiah Institute of Technology, Bangalore-54, Affiliated to VTU, Belagavi, Karnataka, INDIA, mmpraveenamaths@gmail.com

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Abstract

In this paper, we study invariant submanifolds of SQ-Sasakian manifolds. We prove that a three dimensional invariant submanifold of SQ-manifold is totally geodesic. Furthermore, we study an invariant submanifold of SQ-Sasakian manifolds satisfying Q(S,∇̃_X α)=0,Q(S,R̃·α)=0,Q(g,C̃·α)=0 and Q(S,C̃·α)=0. Finally, we construct a non-trivial example to verify our results.

Keywords: Invariant submanifold, SQ-Sasakian manifold, totally geodesic.

1. Introduction

A (2n + 1)-dimensional differentiable manifold $\tilde{M}$ is said to be an almost contact metric manifold if there exists a (1, 1) tensor field ϕ, a vector field ξ and a 1-form η on $\tilde{M}$ such that

$$\phi^2 = -X + \eta(X)\xi, \eta(\xi) = 1 \text{ and } g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \qquad (1)$$

for any vector fields X, Y on $\widetilde{M}$. In specific, in an almost contact metric manifold the following conditions hold true from (1.1)

$$\phi\xi = 0, \eta \cdot \phi = 0 \qquad (2)$$

Such a manifold $\widetilde{M}$ is said to be normal if

$$N_\phi(X,Y) = [\phi,\phi](X,Y) + 2d\eta(X,Y)\xi = 0,$$

for any vector fields X, Y on $\widetilde{M}$, where N_ϕ denotes the Nijenhuis tensor of ϕ and $[\phi,\phi](X,Y)$ is given by

$$[\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

An almost contact metric manifold is said to be contact metric manifold if $\Phi = d\eta$. An almost contact metric manifold $\widetilde{M}(\phi,\xi,\eta,g)$ is said to be [2]

$$\begin{cases} (a)\ Sasakian \Leftrightarrow \Phi = d\eta\ and\ (\phi,\xi,\eta,g) is\ normal; \\ (b)\ \text{cosymplectic } \Leftrightarrow d\Phi = d\eta = 0\ and\ (\phi,\xi,\eta,g) is\ normal; \\ (c)\ \text{quasi} - \text{Sasakian} \Leftrightarrow d\Phi = 0\ and\ (\phi,\xi,\eta,g) is\ normal. \end{cases}$$

A 3-dimensional almost contact metric manifold is quasi-Sasakian if and only if

$$\widetilde{\nabla}_X \xi = -\beta\phi X, \qquad (3)$$

for a certain smooth function β on $\widetilde{M}$ such that $\xi\beta = 0$. However, in general the relation (1.3) does not hold in quasi-Sasakian manifold of dimension greater than three. Later, in 1993 Kwon and Kim [6] have shown in an almost contact metric manifold of dimension greater than three the following relations hold:

$$d\Phi = 0, \widetilde{\nabla}_X \xi = -\beta\phi X, \qquad (4)$$

and almost contact structure is normal for a certain smooth function β on $\widetilde{M}$ such that $\xi\beta = 0$. This is a new class of almost contact metric manifolds satisfying (1.4) and such a manifold is said to be special quasi-Sasakian manifold (briefly SQ-Sasakian manifold) [6] and is denoted by $\widetilde{M}^*$. Here the smooth function β is said to be the structure function of the SQ-Sasakian manifold. SQ-Sasakian manifold becomes cosymplectic if $\beta = 0$ and Sasakian if $\beta = 1$.

On a Riemannian manifold $\widetilde{M}^*$, for a $(0,k)$ −type tensor field $T(k \geq 1)$ and a $(0,2)$ −type tensor field B, we denote by $Q(B,T)$ a $(0,k+2)$ −type tensor field [14] defined as follows:

$$Q(B,T)(X_1,X_2,\cdots,X_k;X,Y) = -T\big((X\wedge_B Y)X_1,X_2,\cdots,X_k\big) - -T(X_1,(X\wedge_B Y)X_2,\cdots,X_k) -T(X_1,X_2,\cdots,(X\wedge_B Y)X_k), \quad (5)$$

where $(X\wedge_B Y)$ is defined by

$$(X\wedge_B Y) = B(Y,Z)X - B(X,Z)Y \quad (6)$$

On the other hand in 1981, Bejancu and Papaghuic introduced the notion of invariant submanifold which inherits almost all properties of ambient manifold. Henceforth several geometers studied invariant submanifolds of different ambient manifolds [10, 11-16]. In the present days this theory plays a key role in image processing, computer geometric design, economic modeling as well as in theoretical physics and applied mathematics.

Recently, S.K. Hui and Joydeb Roy [5], studied invariant and anti-invariant submanifolds of SQ-Sasakian manifold and they have shown that invariant submanifolds of SQ-Sasakian manifold is SQ-Sasakian. Further they have obtained some equivalent conditions for invariance. Motivated by studies of the above authors, in this present paper we consider invariant submanifolds of SQ-Sasakian manifolds satisfying $Q\big(S,\widetilde{\nabla}_X\alpha\big) = 0$, $Q\big(S,\tilde{R}\cdot\alpha\big) = 0$, $Q\big(g,\tilde{C}\cdot\alpha\big) = 0$ and $Q\big(S,\tilde{C}\cdot\alpha\big) = 0$.

2. PRELIMINARIES

For a $(2n+1)$ −dimensional Riemannian manifold, the concircular curvature tensor $\tilde{C}$ is defined by

$$\tilde{C}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{\tilde{r}}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y], \quad (7)$$

for any $X,Y,Z \in \chi(\widetilde{M})$.

Let M be a $(2m+1)$-dimensional (m < n) immersed submanifold of a SQ-Sasakian manifold $\widetilde{M}^*$. Let $\chi(M)$ be the Lie algebra of vector fields on M and $\chi^{\perp}(M)$ be the set of all normal vector fields on M. Let r denotes the covariant differentiation in M determined by the induced metric g.Let α be the second fundamental form of M. Then the formulas of Gauss and Weingarten are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \alpha(X,Y), X,Y \in \chi(M), \quad (8)$$

$$\widetilde{\nabla}_X N = -\mathrm{A_N X} + \nabla_X^{\perp} Y,\ N \in \chi^{\perp}(M), \quad (9)$$

where $g(A_N X,Y) = g(\alpha(X,Y),N)$ and $\nabla_X^{\perp}N$ denotes the covariant derivative of a cross section N of the normal bundle $T^{\perp}M$ in the direction of X with respect to the connection in $T^{\perp}M$. A submanifold M of an SQ-Sasakian manifold $\widetilde{M}^*$ is totally geodesic if $\alpha(X,Y) = 0$, for $X,Y \in TM$. The covariant derivative of α is given by

$$\big(\widetilde{\nabla}_X\alpha\big)(Y,Z) = \nabla_X^{\perp}\big(\alpha(Y,Z)\big) - \alpha(\nabla_X Y,Z) - \alpha(Y,\nabla_X Z), \quad (10)$$

for any $X,Y,Z \in TM$. In 1985, Deprez [4] defined the immersion

$$(\tilde{R}(X,Y)\cdot\alpha)(U,V) = R^{\perp}(X,Y)\alpha(U,V) - \alpha(R(X,Y)U,V) - \alpha(U,R(X,Y)V), \quad (11)$$

for all $X,Y,U,V \in TM$. If $\tilde{R}\cdot\alpha = 0$, then M is said to be semi-parallel. A submanifold M of a SQ-Sasakian manifold $\widetilde{M}^*$ is said to be invariant [1] if $\phi(TM) \subset TM$. On an invariant submanifold $\widetilde{M}^*$, it follows that $\xi \in \chi(M)$. In an invariant submanifold of SQ-Sasakian manifold $\widetilde{M}^*$, the following relations hold:

$$\nabla_X\xi = -\beta\phi X, \quad (12)$$

$$\alpha(X,\xi)=0, \tag{13}$$

$$R(X,Y)\xi=(Y\beta)\phi X-(X\beta)\phi Y+\beta^2[\eta(Y)X-\eta(X)Y], \tag{14}$$

$$S(X,\xi)=2m\beta^2\eta(X)-((\phi X)\beta), \tag{15}$$

for any vector field $X,Y\in TM$.

So we can state the following:

Theorem 1 *[5] An invariant submanifold M of a SQ-Sasakian manifold $\widetilde{M}^*$ is a SQ-Sasakian manifold.*

Proposition 2 *Let M be an invariant submanifold of a SQ-Sasakian manifold then, there exist two differentiable orthogonal distributions D and $D^\perp$ on M such that*

$$TM=D\oplus D^\perp\oplus<\xi>, \tag{16}$$

$$where\ \phi(D)\subset D^\perp,\ \phi(D^\perp)\subset D.$$

Proof. We know that an invariant submanifold of a SQ-Sasakian manifold is also SQ-Sasakian manifold and hence admits almost contact structure. As a consequence the dimension of the invariant submanifold is odd. Let M be an invariant submanifold of a SQ-Sasakian manifold and ξ is tangent to M. Hence one can write $TM=\mathfrak{D}\oplus<\xi>$, where $\mathfrak{D}$ is the orthogonal complement of ξ. Let $X_1\in\mathfrak{D}$. Now from (1) and (2) we have $g(X_1,\phi X_1)=0$ and $g(\xi,\phi X_1)=0$. This implies that ϕX_1 is orthogonal to X_1 and ξ. As a consequence it is possible to write $\mathfrak{D}=D+D^\perp$, where $X_1\in D\subset\mathfrak{D}$ and $\phi X_1\in D^\perp\subset\mathfrak{D}$. For $\phi X_1\in D^\perp$, we have

$$\phi(\phi X_1)=\phi^2X_1=-X_1+\eta(X_1)\xi=-X_1\in D.$$

Hence for $X_1\in D,\phi X_1\in D^\perp$ and for $X_2\in D^\perp,\phi X_2\in D$. This completes the proof of the theorem.

Theorem 3 *Every three dimensional invariant submanifold of a SQ-Sasakian manifold is totally geodesic.*

Proof. Let M be a three dimensional invariant submanifold of a SQ-Sasakian manifold $\widetilde{M}^*$. Then by Proposition (2), there are two orthogonal distributions D and $D^\perp$ satisfying (16).

Let $U_1,V_1\in D$ and $\phi U_1,\phi V_1\in D^\perp$. By using (1) and (16) we get

$$\alpha(\phi U_1,\phi V_1)=-\alpha(U_1,V_1).$$

Let $\phi U_1=U_2$ and $\phi V_1=V_2$, this implies that $U_2,V_2\in D^\perp$. Therefore

$$\alpha(U_2,V_2)=-\alpha(U_1,V_1). \tag{17}$$

By the bilinearity property of α we have

$$\alpha(U_1+U_2+\xi,V_1)=\alpha(U_1,V_1)+\alpha(U_2,V_1)+\alpha(\xi,V_1), \tag{18}$$

$$\alpha(U_1+U_2+\xi,V_2)=\alpha(U_1,V_2)+\alpha(U_2,V_2)+\alpha(\xi,V_2), \tag{19}$$

$$\alpha(U_1+U_2+\xi,\xi)=\alpha(U_1,\xi)+\alpha(U_2,\xi)+\alpha(\xi,\xi). \tag{20}$$

Adding (18), (19) and (20) and by virtue of (13) and (17), we get

$$\alpha(U_1+U_2+\xi,V_1+V_2+\xi)=\alpha(U_2,V_1)+\alpha(U_1,V_2)=0.$$

Since from Proposition (2) we have $TM=D\oplus D^\perp\oplus<\xi>$. So $X,Y\in TM$ can be expressed as $X=U_1+U_2+\xi$ and $Y=V_1+V_2+\xi$. Thus the above equation yields

$$\alpha(X,Y)=0$$

Therefore the submanifold M is totally geodesic.

3. INVARIANT SUBMANIFOLDS OF SQ-SASAKIAN MANIFOLDS SATISFYING $Q(S,\widetilde{\nabla}\alpha)=0$

In this section we study invariant submanifolds of SQ-Sasakian manifolds satisfying $Q(S,\widetilde{\nabla}\alpha)=0$. i.e.,

$$Q(S,\widetilde{\nabla}_X\alpha)(Y,Z;U,V)=0. \tag{21}$$

Making use of relation (5) in (21), we have

$$-(\widetilde{\nabla}_X\alpha)((U\wedge_S V)Y,Z)-(\widetilde{\nabla}_X\alpha)(Y,(U\wedge_S V)Z)=0.$$

Then by the use of (6) we find

$$0=-(\widetilde{\nabla}_X\alpha)(S(V,Y)U,Z)+(\widetilde{\nabla}_X\alpha)(S(U,Y)V,Z)$$
$$-(\widetilde{\nabla}_X\alpha)(Y,S(V,Z)U)+(\widetilde{\nabla}_X\alpha)(Y,S(U,Z)V). \tag{22}$$

By using the equation (10) in (23), we have

$$0=-\nabla_X^{\perp}\alpha(S(V,Y)U,Z)+\alpha(\nabla_X S(V,Y)U,Z)+\alpha(S(V,Y)U,\nabla_X Z)$$
$$+\nabla_X^{\perp}\alpha(S(U,Y)V,Z)-\alpha(\nabla_X S(U,Y)V,Z)-\alpha(S(U,Y)V,\nabla_X Z)$$
$$-\nabla_X^{\perp}\alpha(Y,S(V,Z)U)+\alpha(\nabla_X Y,S(V,Z)U)+\alpha(Y,\nabla_X S(V,Z)U)$$
$$+\nabla_X^{\perp}\alpha(Y,S(U,Z)V)-\alpha(\nabla_X Y,S(U,Z)V)-\alpha(Y,\nabla_X S(U,Z)V).$$

Putting $Y=Z=V=\xi$ and by virtue of (13) we have

$$0=2S(\xi,\xi)\alpha(U,\nabla_X\xi). \tag{23}$$

Making use of (12) and (15) in (23) we have

$$0=4m\beta^3\alpha(U,\phi X).$$

Replacing X by ϕX and making use of (1) and (13), we have $0=4m\beta^3\alpha(U,X)$. Conversely, let M be totally geodesic with $\beta\neq 0$, then from (23) we get $Q(S,\widetilde{\nabla}\alpha)=0$. Thus, we can state the following:

Theorem 4 *An invariant submanifold of a non-cosymplectic SQ-Sasakian manifold satisfies* $Q(S,\widetilde{\nabla}\alpha)=0$ *if and only if it is totally geodesic.*

4. INVARIANT SUBMANIFOLDS OF SQ-SASAKIAN MANIFOLD SATISFYING $Q(S,\widetilde{R}\cdot\alpha)=0$

This section deals with the study of invariant submanifolds of SQ-Sasakian manifolds satisfying $Q(S,\tilde{R}\cdot\alpha)=0$. i.e.,

$$Q(S,\tilde{R}(X,Y)\cdot\alpha)(Z,W;U,V)=0.$$

In view of (5), the above equation can be written as

$$-(\tilde{R}(X,Y)\cdot\alpha)((U\wedge_S V)Z,W)-(\tilde{R}(X,Y)\cdot\alpha)(Z,(U\wedge_S V)W)=0. \tag{24}$$

Making use of relation (6) in (24) we obtain

$$0=-S(V,Z)(\tilde{R}(X,Y)\cdot\alpha)(U,W)+S(U,Z)(\tilde{R}(X,Y)\cdot\alpha)(V,W)$$
$$-S(V,W)(\tilde{R}(X,Y)\cdot\alpha)(Z,U)+S(U,W)(\tilde{R}(X,Y)\cdot\alpha)(Z,V). \tag{25}$$

Using (11) in (25), we have

$$0=-S(V,Z)[R^{\perp}(X,Y)\alpha(U,W)+\alpha(R(X,Y)U,W)+\alpha(U,R(X,Y)W)$$
$$+S(U,Z)[R^{\perp}(X,Y)\alpha(V,W)-\alpha(R(X,Y)V,W)-\alpha(V,R(X,Y)W)$$
$$-S(V,W)[R^{\perp}(X,Y)\alpha(Z,U)+\alpha(R(X,Y)Z,U)+\alpha(Z,R(X,Y)U)$$
$$+S(U,W)[R^{\perp}(X,Y)\alpha(Z,V)-\alpha(R(X,Y)Z,V)-\alpha(Z,R(X,Y)V).$$

Putting $Y=Z=W=V=\xi$ and in view of (13), we get

$$0=S(\xi,\xi)\alpha(U,R(X,\xi)\xi). \tag{26}$$

Taking account of (14), (15) in (26), and using (13), we obtain

$$0=2m\beta^4\alpha(U,X)=0.$$

Hence M is totally geodesic provided $\beta \neq 0$. Conversely, let M be totally geodesic with $\beta \neq 0$, then from (25) we get $Q(S, \tilde{R} \cdot \alpha) = 0$. Hence we can state the following:

Theorem 5 *An invariant submanifold of a non cosymplectic SQ-Sasakian manifold satisfies $Q(S, \tilde{R} \cdot \alpha) = 0$ if and only if it is totally geodesic.*

5. INVARIANT SUBMANIFOLDS OF SQ-SASAKIAN MANIFOLD SATISFYING $Q(g, \widetilde{C} \cdot \alpha) = 0$

This section deals with the study of invariant submanifolds of SQ-Sasakian manifolds satisfying $Q(g, \tilde{C} \cdot \alpha) = 0$. i.e.,

$$Q(g, \tilde{C}(X,Y) \cdot \alpha)(Z, W; U, V) = 0$$

Applying (5), we have

$$-(\tilde{C}(X,Y) \cdot \alpha)((U \wedge_g V)Z, W) - (\tilde{C}(X,Y) \cdot \alpha)(Z, (U \wedge_g V)W) = 0. \tag{27}$$

Making use of relation (14) in (27) we obtain

$$0 = -g(V,Z)(\tilde{C}(X,Y) \cdot \alpha)(U,W) + g(U,Z)(\tilde{C}(X,Y) \cdot \alpha)(V,W)$$
$$-g(V,W)(\tilde{C}(X,Y) \cdot \alpha)(Z,U) + g(U,W)(\tilde{C}(X,Y) \cdot \alpha)(Z,V). \tag{28}$$

By the use of (11) in (28), we have

$$0 = -g(V,Z)[C^{\perp}(X,Y)\alpha(U,W) + \alpha(C(X,Y)U,W) + \alpha(U, C(X,Y)W)$$
$$+g(U,Z)[C^{\perp}(X,Y)\alpha(V,W) - \alpha(C(X,Y)V,W) - \alpha(V, C(X,Y)W)$$
$$-g(V,W)[C^{\perp}(X,Y)\alpha(Z,U) + \alpha(C(X,Y)Z,U) + \alpha(Z, C(X,Y)U)$$
$$+g(U,W)[C^{\perp}(X,Y)\alpha(Z,V) - \alpha(C(X,Y)Z,V) - \alpha(Z, C(X,Y)V).$$

Putting $Y = Z = W = V = \xi$ and in view of (13), one can get

$$0 = g(\xi,\xi)\alpha(U, C(X,\xi)\xi). \tag{29}$$

Taking account of (7), (15) in (29), and using (13), we obtain

$$0 = \left[\beta^2 - \frac{\tilde{r}}{2m(2m+1)}\right]\alpha(U,X) = 0.$$

This implies $\alpha(U,X) = 0$ provided $\tilde{r} \neq 2m(2m+1)\beta^2$. Hence M is totally geodesic provided $\tilde{r} \neq 2m(2m+1)\beta^2$. Conversely, let M be totally geodesic with $\tilde{r} \neq 2m(2m+1)\beta^2$, then from (28) we get $Q(g, C \cdot \alpha) = 0$. Hence we can state the following:

Theorem 6 *An invariant submanifold of a non-cosymplectic SQ-Sasakian manifold with $\tilde{r} \neq 2m(2m+1)\beta^2$ satisfies $Q(g, C \cdot \alpha) = 0$ if and only if it is totally geodesic.*

6. INVARIANT SUBMANIFOLDS OF SQ-SASAKIAN MANIFOLD SATISFYING $Q(S, \widetilde{C} \cdot \alpha) = 0$

This section deals with the study of invariant submanifolds of SQ-Sasakian manifolds satisfying $Q(S, \tilde{C} \cdot \alpha) = 0$. i.e.,

$$Q(S, \tilde{C}(X,Y) \cdot \alpha)(Z, W; U, V) = 0.$$

Using (5), we have

$$-(\tilde{C}(X,Y) \cdot \alpha)((U \wedge_S V)Z, W) - (\tilde{C}(X,Y) \cdot \alpha)(Z, (U \wedge_S V)W) = 0. \tag{30}$$

Making use of relation (6) in (30) we obtain

$$0 = -S(V,Z)(\tilde{C}(X,Y) \cdot \alpha)(U,W) + S(U,Z)(\tilde{C}(X,Y) \cdot \alpha)(V,W)$$
$$-S(V,W)(\tilde{C}(X,Y) \cdot \alpha)(Z,U) + S(U,W)(\tilde{C}(X,Y) \cdot \alpha)(Z,V). \tag{31}$$

By the use of (11) in (31), we have

$$0 = -S(V,Z)[C^{\perp}(X,Y)\alpha(U,W) + \alpha(C(X,Y)U,W) + \alpha(U, C(X,Y)W)$$

$$+S(U,Z)[C^{\perp}(X,Y)\alpha(V,W)-\alpha(C(X,Y)V,W)-\alpha(V,C(X,Y)W)$$
$$-S(V,W)[C^{\perp}(X,Y)\alpha(Z,U)+\alpha(C(X,Y)Z,U)+\alpha(Z,C(X,Y)U)$$
$$+S(U,W)[C^{\perp}(X,Y)\alpha(Z,V)-\alpha(C(X,Y)Z,V)-\alpha(Z,C(X,Y)V).$$

Putting $Y=Z=W=V=\xi$ and in view of (13), one can get

$$0=S(\xi,\xi)\alpha(U,C(X,\xi)\xi,U). \tag{32}$$

Taking account of (7), (15) in (32), and using (13), we obtain

$$0=2m\beta^2\left[\beta^2-\frac{\tilde{r}}{2m(2m+1)}\right]\alpha(U,X)=0.$$

This implies $\alpha(U,X)=0$ provided $\beta\neq 0$ and $\tilde{r}\neq 2m(2m+1)\beta^2$. Hence M is totally geodesic provided $\beta\neq 0$ and $\tilde{r}\neq 2m(2m+1)\beta^2$. Conversely, let M be totally geodesic with $\beta\neq 0$ and $\tilde{r}\neq 2m(2m+1)\beta^2$, then from (31) we get $Q(S,\tilde{C}\cdot\alpha)=0$. Hence we can state the following:

Theorem 7 *An invariant submanifold of a non-cosymplectic SQ-Sasakian manifold with $\tilde{r}\neq 2m(2m+1)\beta^2$ satisfies $Q(S,\tilde{C}\cdot\alpha)=0$ if and only if it is totally geodesic.*

7. EXAMPLE

In this section, we would like to construct an example of a five dimensional SQ-Sasakian manifold.

Let us consider the five-dimensional manifold $\widetilde{M}^*=\{(x_1,x_2,y_1,y_2,z)\in R^5:(x_1,x_2,y_1,y_2,z)\neq 0\}$ where (x_1,x_2,y_1,y_2,z) are the standard coordinates in R^5. The vector fields

$$e_1=\frac{\partial}{\partial x_1}-x_2\frac{\partial}{\partial z},e_2=\frac{\partial}{\partial x_2},e_3=\frac{\partial}{\partial y_1}-y_2\frac{\partial}{\partial z},e_4=\frac{\partial}{\partial y_2},e_5=\frac{\partial}{\partial z},$$

are linearly independent at each point of $\widetilde{M}^*$. Let g be the Riemannian metric defined by

$$g(e_i,e_j)=\begin{cases}1, & if\ \ i=j;\\ 0, & if\ \ i\neq\ \ 0.\end{cases}$$

Let η be the 1-form defined by $\eta(X)=g(X,e_5)$ for any vector field X on $\widetilde{M}^*$. Let ϕ be the (1,1) tensor field defined by $\phi e_1=-e_2$, $\phi e_2=e_1$, $\phi e_3=-e_4$, $\phi e_4=e_3$, $\phi e_5=0$. Then by using the linearity of ϕ and g we have

$$\eta(e_5)=1,\phi^2X=-X+\eta(X)e_5,g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),$$

or any vector fields X,Y on $\widetilde{M}^*$. Thus for $e_5=\xi$ $\widetilde{M}^*(\phi,\xi,\eta,g)$ defines an almost contact metric manifold. Let $\widetilde{\nabla}$ be the Levi-Civita connection with respect to the Riemannian metric g. Then we have

$$[e_1,e_5]=e_2,[e_2,e_5]=e_1,[e_3,e_5]=e_4,[e_4,e_5]=e_3,$$

and remaining $[e_i,e_j]=0$ for all $1\leq i,j\leq 5$. The Levi-Civita connection $\widetilde{\nabla}$ of the metric tensor g is given by Koszul's formula

$$2g(\widetilde{\nabla}_XY,Z)=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)$$
$$-g(X,[Y,Z])-g(Y,[Z,X])+g(Z,[X,Y]).$$

Using Koszul's formula it can be easily calculated that

$$\widetilde{\nabla}_{e_1}e_2=\frac{1}{2}e_5,\ \widetilde{\nabla}_{e_1}e_5=\frac{1}{2}e_2,\ \widetilde{\nabla}_{e_2}e_1=-\frac{1}{2}e_5,\ \widetilde{\nabla}_{e_2}e_5=-\frac{1}{2}e_1,$$
$$\widetilde{\nabla}_{e_3}e_4=\frac{1}{2}e_5,\ \widetilde{\nabla}_{e_3}e_5=\frac{1}{2}e_4,\ \widetilde{\nabla}_{e_4}e_3=-\frac{1}{2}e_5,\ \widetilde{\nabla}_{e_4}e_5=-\frac{1}{2}e_3,$$
$$\widetilde{\nabla}_{e_5}e_1=-\frac{1}{2}e_2,\ \widetilde{\nabla}_{e_5}e_2=\frac{1}{2}e_1,\ \widetilde{\nabla}_{e_5}e_3=-\frac{1}{2}e_4,\ \widetilde{\nabla}_{e_5}e_4=\frac{1}{2}e_3,$$
$$\widetilde{\nabla}_{e_i}e_j=0, otherwise.$$

We have seen that the (ϕ,ξ,η,g) structure satisfies the formula $\tilde{\nabla}_X\xi = -\beta\phi X$. Hence $\tilde{M}(\phi,\xi,\eta,g)$ is a SQ-Sasakian manifold with the structure function $\beta = -\frac{1}{2}$. Let M be a subset of $\tilde{M}$ and consider the isometric immersion $f: M \to \tilde{M}$ defined by $f(x_1,x_2,z) = f(x_1,x_2,0,0,z)$.

It can be easy to prove that $M = \{(x_1,x_2,z) \in R^3 : (x_1,x_2,z) \neq 0\}$, where (x_1,x_2,z) are standard coordinates of R^3 is a 3-dimensional submanifold of the 5-dimensional SQ-Sasakian manifold $\tilde{M}$. We choose the vector fields

$$e_1 = \frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial z}, e_2 = \frac{\partial}{\partial x_2}, e_5 = \frac{\partial}{\partial z},$$

which are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_i,e_j) = \begin{cases} 1, & if\ \ i = j; \\ 0, & if\ \ i \neq\ \ 0. \end{cases}$$

where i and j takes the values 1,2,5.

We define 1-form η and (1,1)-tensor ϕ respectively by

$$\eta(X) = g(X,e_5), \phi e_1 = -e_2, \phi e_2 = e_1, \phi e_5 = 0.$$

The linear property of g and ϕ shows that

$$\eta(e_5) = 1, \phi^2 X = -X + \eta(X)e_5, g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for all vector fields X, Y on M. Keeping $e_5 = \xi$ and using Koszul's formula for the metric g, it can be easily find that

$$\nabla_{e_1}e_2 = \frac{1}{2}e_5,\ \nabla_{e_1}e_5 = \frac{1}{2}e_2,\ \nabla_{e_2}e_1 = -\frac{1}{2}e_5,\ \nabla_{e_2}e_5 = -\frac{1}{2}e_1,$$

$$\nabla_{e_5}e_1 = -\frac{1}{2}e_2,\ \nabla_{e_5}e_2 = \frac{1}{2}e_1,\ \nabla_{e_i}e_j = 0, otherwise.$$

Let us take

$$TM = D \oplus D^{\perp} \oplus < \xi >,$$

where $D =< e_1 >, D^{\perp} =< e_2 >$. Then we observe that $\phi e_1 = -e_2 \in D^{\perp}$, for $e_1 \in D$ and $\phi e_2 = e_1 \in D$, for $e_2 \in D^{\perp}$. Hence the submanifold is invariant and also Proposition (2) is verified. Now form the values of $\tilde{\nabla}_{e_i}e_j$ and $\nabla_{e_i}e_j$ and the relation $\alpha(e_i,e_j) = \tilde{\nabla}_{e_i}e_j - \nabla_{e_i}e_j$, we see that $\alpha(e_i,e_j) = 0$. Thus the submanifold is totally geodesic. Hence the Theorem (3) is verified.

8. CONCLUSION

In this paper, the geometric properties of invariant submanifolds of SQ-Sasakian manifolds are studied. We gave necessary and sufficient geometric conditions for invariant submanifolds of SQ-Sasakian manifolds to be totally geodesic. An SQ-Sasakian manifold is a contact manifold which is a generalization of Sasakian and cosymplectic manifolds. Hence our study is important due to applications of invariant submanifolds in image processing, computer geometric design and economic modelling and applications of contact geometryin control theory, quantum mechanics and thermodynamics. By virtue of Theorems (4), (5), (6) and (7), we can state the following: From the Theorems we can state the following result:

Theorem 8 *Let M be an invariant submanifold of a non-cosymplectic SQ-Sasakian manifold. Then, the following statements are equivalent:*

1. M is totally geodesic;
2. M satisfies $Q(S,\tilde{\nabla}\alpha) = 0$;
3. M satisfies $Q(S,\tilde{R}\cdot\alpha) = 0$;
4. M satisfies $Q(S,\tilde{C}\cdot\alpha) = 0$ provided $\tilde{r} \neq 2m(2m+1)\beta^2$;
5. M satisfies $Q(g,\tilde{C}\cdot\alpha) = 0$ provided $\tilde{r} \neq 2m(2m+1)\beta^2$.

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